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PERTURBATIONS CAUSING OSCILLATIONS OF FUNCTIONAL-DIFFERENTIAL EQUATIONS

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ABSTRACT. Some new criteria are given for the oscillation of solutions of perturbed functional-differential equations of the form

$$x^{(n)} + P(t)f(x(g(t))) = Q(t). \tag{I}$$

The results are new even in the case $g(t) \equiv t$, or when (I) is linear. The function $Q(t)$ does not have to be small or periodic.

1. **Introduction.** The first of the authors raised in [4] the following question: What kind of perturbations $Q(t)$ force all solutions of the equation

$$x^{(n)} + P(t, x, x', \cdots, x^{(n-1)}) = Q(t), \qquad n \geqq 2, \tag{*}$$

to oscillate, although the homogeneous equation is not necessarily oscillatory? It is our intention here to answer this question for a large class of equations. Actually, we present our result in the case of a functional-differential equation of the type

$$x^{(n)} + P(t)f(x(g(t))) = Q(t). \tag{I}$$

The theorems of this paper are entirely new even in the case $g(t) \equiv t$, or when f is linear.

Possible generalizations and some examples are discussed at the end of the paper. For results related to the contents of this paper the reader is referred to Kartsatos [2], [3], [4], Teufel [6], Atkinson [1] for the case of an ordinary equation, and True [7], Kusano and Onose [5] for the case (I).

In what follows, use will be made of the following conditions:

(i) $P \in C[[0, \infty), \boldsymbol{R}]$, $\boldsymbol{R} = (-\infty, \infty)$;

(ii) $g \in C[[0, \infty), \boldsymbol{R}]$, $\lim_{t\to\infty} g(t) = +\infty$;

(iii) $f \in C[\boldsymbol{R}, \boldsymbol{R}]$, increasing and $uf(u) > 0$ for $u \in R$ with $u \neq 0$;

(iv) $Q \in C[[0, \infty), \boldsymbol{R}]$.

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By a solution of (I) we mean a function $x \in C^n[[t_x, \infty), \boldsymbol{R}]$, which satisfies (I) for all $t \in [t_x, \infty)$. Here $t_x \geqq 0$ depends on the particular solution $x(t)$. Let $\mathscr{F}$ denote the family of all such solutions of (I). A function $x \in \mathscr{F}$ is said to be "oscillatory" if it has an unbounded set of zeros in $[t_x, \infty)$. Equation (I) is called oscillatory if every $x \in \mathscr{F}$ is oscillatory.

2. **Main results.** The following theorem ensures the oscillation of all solutions of equation (I).

THEOREM 1. *Along with the hypotheses* (i)–(iv), *assume that* $P(t) \geqq 0$, $t \in [0, \infty)$ *and that there exists a function* $R \in C^n[0, \infty)$, *oscillatory and, such that* $R^{(n)}(t) = Q(t)$, $t \in [0, \infty)$. *Moreover, assume that for every* $\lambda > 0$,

$$\limsup_{t \to \infty} \int_0^t P(s) f(\lambda + R(g(s)))\, ds = +\infty,$$

$$\liminf_{t \to \infty} \int_0^t P(s) f(-\lambda + R(g(s)))\, ds = -\infty.$$

Then for n even, equation (I) *is oscillatory. For n odd every* $x \in \mathscr{F}$ *is oscillatory, or such that* $\lim[x(t) - R(t)] = 0$ *monotonically as* $t \to \infty$. *If the above integral conditions hold for* $\lambda = 0$, *then* (I) *is oscillatory also for n odd.*

PROOF. Assume that the above integral conditions hold for every $\lambda > 0$ and that n is even. Assume that (I) is not oscillatory. Let $x \in \mathscr{F}$ be such that $x(t) > 0$, $t \geqq t_0 \geqq t_x$. Let $u(t) = x(t) - R(t)$, $t \in [t_0, \infty)$. Then $u(t)$ satisfies the equation

$$u^{(n)} + P(t) f(u(g(t)) + R(g(t))) = 0. \tag{1}$$

Since $u(t) + R(t) > 0$ for $t \in [t_0, \infty)$, it follows that there exists $t_1 \geqq t_0$ such that $u(g(t)) + R(g(t)) > 0$ for $t \geqq t_1$. Thus,

$$u^{(n)}(t) = -P(t) f(u(g(t)) + R(g(t))) \leqq 0, \qquad t \geqq t_1. \tag{2}$$

Consequently, all the derivatives of $u(t)$ up to the order n are eventually of one sign and none of them is identically zero on an infinite interval, because this would imply the same for the derivative of order n, a contradiction to the first of the integral assumptions. Furthermore, $u(t)$ is strictly increasing, or strictly decreasing for all large t. Assume for the moment that $u(t) < 0$ for all large t. Then there exists $t_2 \geqq t_1$ such that $u(g(t)) < 0$ for $t \geqq t_2$. Thus, $0 > u(g(t)) > -R(g(t))$, a contradiction to the oscillatory character of $R(t)$. It follows that $u(t) > 0$ for all large t, and this implies (cf. also Kartsatos [3, Theorem 2]) that $u'(t) > 0$ and $u^{(n-1)}(t) > 0$ for all large t, Thus, there exists $t_2 \geqq t_1$ such that

$$u^{(n-1)}(t) > 0 \quad \text{and} \quad u(g(t)) \geqq \lambda > 0 \quad \text{for } t \geqq t_2, \tag{3}$$

where λ is a constant. Integrating (1) we obtain, for $t \geqq t_2$,

$$
\begin{aligned}
u^{(n-1)}(t) &= u^{(n-1)}(t_2) - \int_{t_2}^{t} P(s) f(u(g(s)) + R(g(s)))\, ds \\
&\leqq u^{(n-1)}(t_2) - \int_{t_2}^{t} P(s) f(\lambda + R(g(s)))\, ds.
\end{aligned} \tag{4}
$$

This easily implies $\liminf_{t\to\infty} u^{(n-1)}(t) = -\infty$, a contradiction to the first of (3). Consequently, $u(t)$ cannot be positive, or negative, or oscillatory for all large t, i.e., $u(t)$ does not exist under the assumption $x(t)>0$, $t \geqq t_0$. A similar situation appears in the case $x(t)<0$, $t \geqq t_0$, and this completes the proof of the theorem in the case of even n. Exactly the same argument applies in the case of odd n, except the fact that $u'(t)$ could be eventually negative when $u(t)>0$ for all large t, or eventually positive when $u(t)<0$ for all large t. Both cases imply $\lim u(t) = \lim [x(t) - R(t)] = 0$ monotonically as $t \to \infty$. For such considerations the reader is referred to the proof of Theorem 1 in Kartsatos [**4**]. Now if the integral conditions hold for $\lambda = 0$, then they hold for all $\lambda > 0$, and the theorem is true for the case of even n. However, if n is odd we obtain from the equation in (4)

$$
u^{(n-1)}(t) \leqq u^{(n-1)}(t_2) - \int_{t_2}^{t} P(s) f(R(g(s)))\, ds, \tag{5}
$$

for all $t \geqq$ (some) t_2. This implies $\lim_{t\to\infty} u^{(n-1)}(t) = -\infty$, or $\lim_{t\to\infty} u(t) = -\infty$, a contradiction to the positivity of $u(t)$. An analogous situation holds in the case of an eventually negative $u(t)$. Consequently, $x(t)$ is oscillatory, and the proof is complete.

We now give two useful corollaries to the above theorem.

COROLLARY 1. *Assume that $f(u) = u^{2q+1}$, where q is a nonnegative integer. Assume further that the hypotheses* (i)–(iv) *are satisfied, $P(t) \geqq 0$ for $t \geqq 0$, and Q, R are as in Theorem* 1. *Moreover, let*

$$
\int_0^\infty P(t)\, dt = +\infty, \qquad \int_0^\infty P(t)\, |R(g(t))|^m\, dt < \infty, \tag{6}
$$

for every $m = 1, 2, \cdots, 2q+1$. Then the conclusion of Theorem 1 *concerning the case $\lambda \neq 0$ is satisfied.*

PROOF. It suffices to observe that, for a real $\lambda \neq 0$,

$$
(\lambda + R(g(t)))^{2q+1} = \sum_{k=0}^{2q+1} \binom{2q+1}{k} \lambda^{2q+1-k} [R(g(t))]^k. \tag{7}
$$

COROLLARY 2. *Let the assumptions of Corollary* 1 *be satisfied except* (6). *Let*

$$\limsup_{t\to\infty}\int_0^\infty P(t)[R(g(t))]^m\,dt = +\infty,$$

$$\liminf_{t\to\infty}\int_0^\infty P(t)[R(g(t))]^m\,dt = -\infty$$

for some even integer m with $2\leqq m\leqq 2q$, *and*

$$\int_0^\infty P(t)\,|R(g(t))|^j\,dt < +\infty$$

for each $j=0, 1, 2, \cdots, m-1, m+1, \cdots, 2q+1$.

Then the conclusion of Theorem 1 *concerning the case* $\lambda\neq 0$ *is satisfied.*

PROOF. Another application of formula (7), since every even power of $R(g(t))$ is multiplied by an odd power of λ.

The effect of "weighted" integral conditions on the bounded solutions of (I) is given by the following

THEOREM 2. *Let the assumptions of Theorem* 1 *hold with R bounded and the integral conditions replaced by*

$$\limsup_{t\to\infty}\int_0^t s^{m_1}P(s)f(\lambda + R(g(s)))\,ds = +\infty, \tag{8}$$

$$\liminf_{t\to\infty}\int_0^t s^{m_2}P(s)f(-\lambda + R(g(s)))\,ds = -\infty, \tag{9}$$

where m_1, m_2 *are integers with* $1\leqq m_1$, $m_2\leqq n-1$.

Then (a) *if* (8), (9) *hold for every* $\lambda>0$, *every bounded* $x\in\mathscr{F}$ *is oscillatory for n even, and every bounded* $x\in\mathscr{F}$ *is oscillatory, or such that* $\lim[x(t)-R(t)]=0$ *monotonically as* $t\to\infty$ *for n odd*; (b) *if* (8), (9) *hold for* $\lambda=0$, *every bounded* $x\in\mathscr{F}$ *is oscillatory.*

PROOF. Assume that n is even and that $x(t)$, $u(t)$ are as in the proof of Theorem 1 up to the inequalities (3). Then consider the function $t^{m_1}u^{(n-1)}(t)$. Differentiation of this function gives

$$[t^{m_1}u^{(n-1)}(t)]' = -t^{m_1}P(t)f(u(g(t)) + R(g(t))) + m_1t^{m_1-1}u^{(n-1)}(t),$$

and integration from t_2 to $t\geqq t_2$ gives

$$\begin{aligned} t^{m_1}u^{(n-1)}(t) = t_2^{m_1}u^{(n-1)}(t_2) &- \int_{t_2}^t s^{m_1}P(s)f(u(g(s)) + R(g(s)))\,ds \\ &+ m_1\int_{t_2}^t s^{m_1-1}u^{(n-1)}(s)\,ds. \end{aligned} \tag{10}$$

This implies (by condition (8))

$$\liminf_{t\to\infty}\left[t^{m_1}u^{(n-1)}(t) - m_1\int_{t_2}^{t} s^{m_1-1}u^{(n-1)}(s)\,ds\right] = -\infty. \tag{11}$$

However, since $u^{(n-1)}(t)>0$ for $t\geqq t_2$, and the integral in (11) is an increasing function of t, we have

$$\lim_{t\to\infty}\int_{t_2}^{t} s^{m_1-1}u^{(n-1)}(s)\,ds = +\infty. \tag{12}$$

The proof now continues as in Theorem 1 of Kartsatos [3]. If $x(t)$ is bounded then $u(t)$ is also bounded and $(-1)^k u^{(k)}(t)<0$ for $k=1, 2, \cdots, n-1$ and $t\in[t_2, \infty)$, and successive integration by parts of the integrand in (12) yields eventually

$$\begin{aligned}\lim_{t\to\infty}\int_{t_2}^{t} u^{(n-m)}(s)\,ds &= +\infty, \quad \text{if } m \text{ is odd,}\\ &= -\infty, \quad \text{if } m \text{ is even,}\end{aligned}$$

which is a contradiction to the boundedness of $u(t)$:

$$\lim_{t\to\infty}[u^{(n-m-1)}(t) - u^{(n-m-1)}(t_2)] = \pm\infty.$$

In order to avoid repetition, we omit the rest of the proof concerned with the case of eventually negative $u(t)$, or n odd, or $\lambda=0$.

It is natural to expect now that similar results hold in the case of a nonpositive $P(t)$. As the following example indicates, this is not always the case.

EXAMPLE 1. Consider the second order equation

$$x'' - (e^t - Q(t))e^{-t}x = Q(t), \tag{13}$$

with $R(t)=e^{t/2}\sin t$ and $Q(t)=R''(t)$. Then it is easy to check that the integral conditions of Theorem 1 hold for $\lambda=0$. However, (13) has the nonoscillatory solution $x(t)=e^t$.

Nevertheless, there are some cases of importance where Theorems 1 and 2 have analogues. The following theorem ensures the oscillation of all bounded solutions of the equation

$$x^{(n)} - P(t)f(g(t)) = Q(t) \tag{II}$$

with $P(t)\geqq 0$.

THEOREM 3. *Theorem 2 holds for the equation* (II), *if m_1, m_2 in* (8), (9) *are allowed to be zero, and if "odd" is replaced everywhere by "even" and conversely.*

PROOF. Assume that $x(t)$, $t \in [t_x, \infty)$, is a bounded nonoscillatory solution of (II). Then, without loss of generality, we assume that $x(t)>0$, $t \in [t_1, \infty)$, $t_1 \geqq t_x$. Then $u(t)=x(t)-R(t)$ is a bounded solution of the equation

$$u^{(n)} - P(t)f(u(g(t)) + R(g(t))) = 0, \tag{14}$$

for which $u(g(t))+R(g(t))>0$ for every $t \geqq t_2 \geqq t_1$. Consequently, $u^{(n)}(t) \geqq 0$ for every $t \geqq t_2$, which implies that all the derivatives $u^{(i)}(t)$, $i=0, 1, \cdots, n$, are of fixed sign for all large t, and no two consecutive derivatives are of the same sign for all large t, for this would force $u(t)$ to diverge to $\pm\infty$ as $t \to \infty$. Thus, $u^{(n-1)}(t) \leqq 0$ for $t \geqq t_3 \geqq t_2$. Now assume that the integral condition (8) holds for every $\lambda>0$. Then, as in the proof of Theorem 2, we obtain

$$\begin{aligned} t^{m_1}u^{(n-1)}(t) &= t_3^{m_1}u^{(n-1)}(t_3) + \int_{t_3}^t s^{m_1}P(s)f(u(g(s)) + R(g(s)))\,ds \\ &\quad + m_1\int_{t_3}^t s^{m_1-1}u^{(n-1)}(s)\,ds, \qquad t \geqq t_3. \end{aligned} \tag{15}$$

Now assume that n=odd. Then since $u'(t) \geqq 0$ for $t \geqq t_3$, $u(g(t)) \geqq \lambda > 0$ for $t \in [t_3, \infty)$ (t_3 can be chosen a priori this way). Thus, from (15) we obtain

$$\begin{aligned} t^{m_1}u^{(n-1)}(t) &- m_1\int_{t_3}^t s^{m_1-1}u^{(n-1)}(s)\,ds \\ &\geqq t_3^{m_1}u^{(n-1)}(t_3) + \int_{t_3}^t s^{m_1}P(s)f(\lambda + R(g(s)))\,ds \end{aligned}$$

for $t \geqq t_3$, which implies $\lim_{t\to\infty} \int_{t_3}^t s^{m_1-1}u^{(n-1)}(s)\,ds = -\infty$, a contradiction as in Theorem 2. We omit the rest of the proof.

3. **Discussion–Examples.** It is possible to extend the present results to equations of the form

$$x^{(n)} + P_0(t, \tilde{x}(t), \tilde{x}(g(t))) = Q(t), \tag{III}$$

where $\tilde{x}(t)=(x(t), x'(t), \cdots, x^{(n-1)}(t))$, under suitable assumptions on the function P_0. For example, P_0 could be considered as bounded above and below by functions of the form $P(t)f(x(g(t)))$. A lot of open problems arise now with respect to the choice of the perturbation $Q(t)$. For example, there exist oscillating perturbations, which cannot be represented as nth derivatives of oscillating functions. For example, the function $Q(t) \equiv \frac{1}{2}+2\sin t$. Any function $R(t)$ with $R^{(n)}(t)=Q(t)$, would have to satisfy $R^{(n-1)}(t)=\frac{1}{2}t-2\cos t+C$ (C constant) for all large t, i.e., $\lim_{t\to\infty} R(t)=\infty$. However, the equation

$$x'' - x = \tfrac{1}{2} + 2\sin t \tag{16}$$

has all its bounded solutions oscillatory. In fact, the general solution of (16) is $x(t)=c_1e^t+c_2e^{-t}-\sin t-\frac{1}{2}$. Consequently, there are large classes of equations to be studied under the above considerations. In short, it should be possible to study the oscillatory character of (II) without using the transformation $u(t)=x(t)-R(t)$. Another open problem is the following: Assuming that the homogeneous equation is oscillatory, what kind of perturbations $Q(t)$ stop the oscillation of all, or part of the solutions of (II)?

In view of Corollaries 1 and 2 it is easy to give examples of equations satisfying all the assumptions of Theorem 1. An equation satisfying all the assumptions of Theorem 2 is the following

$$x'' + [\tfrac{1}{4}(1 + t^2)]x = [t^{-\alpha} \sin t]'', \qquad t \geqq 1, \tag{17}$$

where α is a constant with $0<\alpha<1$. It follows that every solution of (17) is either unbounded or bounded and oscillatory.

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