PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 43, Number 1, March 1974

AN EXAMPLE CONCERNING CONTINUOUS SELECTIONS ON INFINITE-DIMENSIONAL SPACES[1]

CARL P. PIXLEY[2]

ABSTRACT. This paper shows that the conclusion of E. Michael's Selection Theorem 1.2 [*Continuous selections*. II, Ann. of Math. (2) **64** (1956), 562–580. MR **18**, 325.] does not necessarily hold if the hypotheses are modified by dropping the assumption that the domain of the lsc carrier is of finite dimension, and strengthening the equilocal and global mapping conditions on the collection of images under the carrier.

1. **Introduction.** This paper investigates one possibility of extending the selection theory for finite-dimensional spaces presented in Michael [**6**] to infinite-dimensional spaces. Let X and Y be topological spaces and let ϕ be a function (called a *carrier*) whose domain is X and whose range is a subset $\mathscr{S}$ of 2^Y (the collection of nonempty closed subsets of Y). The selection problem is the following. What conditions on X, Y, ϕ and $\mathscr{S}$ will guarantee the existence of a continuous function $f\colon X\to Y$ such that $f(x)\in\phi(x)$ for every $x\in X$? The continuous function f is called a *selection* for ϕ.

For reasons given in Michael [**5**], we assume that the carrier ϕ is *lsc* (*lower semicontinuous*), meaning that if U is an open subset of Y, then $\{x\in X\colon\phi(x)\cap U\neq\varnothing\}$ is an open subset of X. The selection theory of Michael [**6**] requires $\mathscr{S}$ to satisfy equilocal mapping conditions commensurate with the dimension of X. The collection $\mathscr{S}\subseteq 2^Y$ is *equi-LC^n* if for every point $y\in\bigcup\mathscr{S}$ and neighborhood U of y in Y, there exists a neighborhood V of y in Y such that, for every $S\in\mathscr{S}$ and $k\leq n$, any continuous map of a k-sphere into $V\cap S$ is null-homotopic in $U\cap S$. A space

Presented to the Society, January 18, 1972; received by the editors February 28, 1972 and, in revised form, March 10, 1973 and June 4, 1973.

AMS (MOS) subject classifications (1970). Primary 54C65; Secondary 54C55, 54C20, 54C60, 57A20, 58B99.

Key words and phrases. Selection, lower semicontinuous, absolute extensor, Hilbert cube, infinite dimensional.

[1] The author wishes to thank Professor Prabir Roy and Mr. John Walsh for introducing Borsuk's example to him. He is also grateful for numerous helpful suggestions of the referee concerning the presentation of this result.

[2] Partial support by the Research Foundation of the State of New York is gratefully acknowledged.

S is C^n if, for every $k \leqq n$, any map of a k-sphere into S is null-homotopic in S. The main result of Michael [**6**] is the following:

THEOREM. *Let X be a paracompact space, $A \subseteq X$ closed with* $\dim X \leqq n+1$,[3] *Y a complete metric space, $\mathcal{S} \subseteq 2^Y$ equi-LC^n, and $\phi: X \to \mathcal{S}$ lsc. Then every selection for $\phi|A$ can be extended to a selection for $\phi|U$ for some open $U \supseteq A$. If, in addition, every $S \in \mathcal{S}$ is C^n, then one can take $U = X$.*

The following question immediately arises. Will the above assertion remain true if we drop the hypothesis that X is finite dimensional and require $\mathcal{S}$ to satisfy very strong equilocal and global extension properties? A metric space S is said to be an *AE* (*absolute extensor*) if, for every metric space X and continuous function f mapping a closed subset A of X into S, there exists a continuous extension of f mapping all of X into S. (*Note*. An *AE* is often called an *AR* (metric).) A collection $\mathcal{S} \subseteq 2^Y$ is *uniformly equi-LAE* (*local absolute extensor*) if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if f is a continuous function from a closed subset A of a metric[4] space X into any $S \in \mathcal{S}$ with $\operatorname{diam}(f(A)) < \delta$, then there exists a continuous extension $\tilde{f}: X \to S$ of f with $\operatorname{diam}(\tilde{f}(X)) < \varepsilon$. Clearly, S is *AE* implies S is C^n for all n, and $\mathcal{S}$ is uniformly equi-*LAE* implies $\mathcal{S}$ is equi-LC^n for all n.

Let Q denote the Hilbert cube, i.e., the set of mappings of the positive integers ω into $[0, 1]$, given the product topology. We use the metric $d(x, y) = \sum_{i=1}^{\infty} |x_i - y_i| \cdot 2^{-i}$. The following result is relevant to the above question.

THEOREM 1.1. *There exists a lsc carrier $\phi: Q \to 2^Q$ such that*

(1) *the collection $\phi(Q)$ is uniformly equi-LAE,*

(2) *if $x \in Q$, then $\phi(x)$ is homeomorphic to a point, or to a k-cell* (*for some* $k \geqq 1$), *or to Q, and*

(3) *there is no selection for ϕ and, in fact, for some $p \in Q$ there is no selection for ϕ restricted to any neighborhood of p.*

To prove this theorem we modify a construction due to K. Borsuk [**1**, pp. 124–127] of a locally contractible compact metric space which is not an *ANR*.

[3] We define $\dim X \leqq n+1$ to mean that every open cover $\mathcal{U}$ of X has an open locally finite refinement $\mathcal{V}$ such that any collection of $n+3$ many distinct elements of $\mathcal{V}$ have no point in common.

[4] If "T_2-normal" is substituted for "metric" here, Theorem 1.1 remains valid. The crucial part, (2) implies (1), of Theorem 3.1 remains valid, if the second occurrence of "metric" in the definition of *ANE* is changed to "T_2-normal".

2. **The carrier ϕ.** In this section we define ϕ. We verify that ϕ is lsc and that condition (2) of Theorem 1.1 holds.

For each integer $k \geqq 2$, let

$$E_k = \left\{ x \in Q : \frac{1}{k+1} \leqq x_1 \leqq \frac{1}{k} \text{ and } x_i = 0 \text{ for } i > k+1 \right\}$$

and let

$$X_k = \left\{ x \in E_k : x_1 \in \left\{ \frac{1}{k+1}, \frac{1}{k} \right\} \text{ or } x_2 = 1 \right.$$
$$\left. \text{or } x_i \in \{0, 1\} \text{ for some } i \text{ with } 2 < i \leqq k+1 \right\}.$$

Notice that $\{x \in X_k : x_2 = 1\}$ is a k-cell, and that, for each r with $0 \leqq r < 1$, the set $\{x \in X_k : x_2 = r\}$ is a $(k-1)$-sphere. Hence X_k is a k-cell. Define $X_\omega = \{x \in Q : x_1 = 0\}$, and let $\tilde{X}_k = E_k - X_k$.

Define $\phi : Q \to 2^Q$ by

$$\begin{aligned} \phi(x) &= \{x\} && \text{if } x \in \bigcup \{X_i : 2 \leqq i\} \cup X_\omega, \\ &= X_k && \text{if } x \in \tilde{X}_k, \\ &= Q && \text{otherwise.} \end{aligned}$$

Condition (2) of Theorem 1.1 holds.

To see that ϕ is lsc, consider $X = \bigcup \{E_i : 2 \leqq i\} \cup X_\omega$. Since X is closed in Q, and $\phi(x) = Q$ for each $x \in Q - X$, it is sufficient to show that $\phi | X$ is lsc. We need to show that if $x_0 \in X$ and $p \in \phi(x_0)$ and p belongs to open set U, then $A = \{x \in X : \phi(x) \cap U \neq \varnothing\}$ contains x_0 as an interior point (in X). We distinguish the three possibilities:

(1) If $p \in X_i - (X_{i-1} \cup X_{i+1})$, then $x_0 \in \tilde{X}_i$ or $x_0 = p$. In either case, x_0 is interior to $\tilde{X}_i \cup (U \cap X_i) \subseteq A$.

(2) If $p \in X_i \cap X_{i+1}$, then $x_0 = p$, or $x_0 \in \tilde{X}_i$, or $x_0 \in \tilde{X}_{i+1}$. In each case, x_0 is interior to $\tilde{X}_i \cup \tilde{X}_{i+1} \cup (U \cap (X_i \cup X_{i+1})) \subseteq A$.

(3) If $p \in X_\omega$, then $p = x_0$. In this case, there exists an N such that $U \cap X_k \neq \varnothing$ for all $k \geqq N$. Therefore x_0 is interior to $U \cap (\bigcup \{E_k : N \leqq k\} \cup X_\omega) \subseteq A$.

The carrier ϕ is therefore lsc.

3. **A characterization of the uniformly equi-*LAE* property.**[5] In this section we obtain a condition equivalent to the condition that a collection $\mathscr{S}$ is uniformly equi-*LAE*. A metric space S is called an *ANE* (*absolute neighborhood extensor*) if, for every metric space X and continuous function

[5] This section was suggested by the referee. His proof of Theorem 3.1 considerably shortened the author's original proof that $\phi(Q)$ is uniformly equi-*LAE*.

f mapping a closed subset A of X into S, there exists a continuous extension of f mapping some neighborhood of A into S. Such a space is frequently called an *ANR* (metric) in the literature. Let Y be a metric space and $\mathscr{S}\subseteq 2^Y$. We define $\mathscr{S}$ to be *uniformly equi-LC* (*locally contractible*) if it is true that for every $\varepsilon>0$ there exists $\delta>0$ such that, if $S\in\mathscr{S}$ and $p\in S$, then $N_\delta(p)\cap S$ is contractible over a subset of S having diameter $<\varepsilon$.

THEOREM 3.1. *Let Y be a metric space and $\mathscr{S}\subseteq 2^Y$. The following conditions are equivalent:*

(1) *The collection $\mathscr{S}$ is uniformly equi-LAE.*

(2) *Each $S\in\mathscr{S}$ is an ANE and $\mathscr{S}$ is uniformly equi-LC.*

PROOF. (1) *implies* (2). Suppose that $\mathscr{S}$ is uniformly equi-*LAE*. If $S\in\mathscr{S}$ then every point of S has a neighborhood (in S) which is an *ANE*. This implies that S is an *ANE* [7, Proposition 4.1].

Now suppose $\varepsilon>0$ is given. Let δ illustrate that $\mathscr{S}$ is uniformly equi-*LAE*. Then $\delta/3$ illustrates the uniformly equi-*LC* property for ε: Let $p\in S\in\mathscr{S}$ and let $A=N_{\delta/3}(p)$. We define $f: A\times\{0,1\}\to S$ by $f(x,0)=x$ and $f(x,1)=p$ for each $x\in A$. Since $A\times\{0,1\}$ is a closed subset of the metric space $A\times[0,1]$, and since $\operatorname{diam}(f(A))\leqq 2\delta/3$, there is a continuous function $\tilde{f}: A\times[0,1]\to S$ such that $\operatorname{diam}(\tilde{f}(A\times[0,1]))<\varepsilon$.

(2) *implies* (1). Let $\varepsilon>0$ be given, and let $\delta>0$ illustrate that $\mathscr{S}$ is uniformly equi-*LC*. Then δ also illustrates that $\mathscr{S}$ is uniformly equi-*LAE*: In fact, suppose that X is metrizable, $A\subseteq X$ closed, $S\in\mathscr{S}$, $f: A\to S$ continuous, and $\operatorname{diam}(f(A))<\delta$. Choose $y\in f(A)$, and let $V=N_\delta(y)\cap S$. Then V is open in S, and therefore an *ANE* (every open subset of an *ANE* is an *ANE*). Therefore f continuously extends to a neighborhood U of A into V. By assumption, V is contractible over a subset Z of S of diameter $<\varepsilon$.[6] Lemma 3.2 (below) insures that f can be extended to a continuous $g: X\to Z$, which finishes the proof of the theorem.

The proof of the following lemma is almost identical to that of the slightly weaker Theorem 12.3 of Hanner [3].

LEMMA 3.2. *Suppose A is a closed subset of normal space X, U is an open neighborhood of A, Y is contractible in space Z, and $f: U\to Y$ is continuous. Then $f|A$ can be extended to a continuous $g: X\to Z$.*

PROOF. Let V be a neighborhood of $X-U$ such that $\bar{V}\cap A=\varnothing$. Let $h: X\to[0,1]$ be continuous such that $h(A)=0$ and $h(\bar{V})=1$. Let $H: Y\times[0,1]\to Z$ illustrate that Y is contractible in Z to a point, say p.

[6] The referee observed that if one applies Borsuk's homotopy extension theorem [1, (8.1), p. 94] to an ε-small open neighborhood of Z, the proof is finished.

Define $g:X\to Z$ by $g(x)=p$ if $x\in \bar{V}$, and $g(x)=H(f(x), h(x))$ if $x \in U$. One easily sees that g is the desired continuous function.

4. **The collection $\phi(Q)$ is uniformly equi-*LAE*.** We verify that $\phi(Q)\subseteq 2^Q$ satisfies condition (2) of Theorem 3.1. Since each $S\in\phi(Q)$ is a point, or the Hilbert cube, or a finite-dimensional cell, it is an *ANE* (in fact, an *AE*). This standard result is a corollary [**1**, IV (7.2), p. 92] of the Tietze extension theorem.

LEMMA 4.1. *The collection $\phi(Q)$ is uniformly equi-LC.*

PROOF. Let $\varepsilon>0$ be given, and choose an integer $m>2$ such that $2^{-m}<\frac{1}{3}\varepsilon$; also choose a positive $\gamma<2^{-(m+1)}$. Let $k\geqq m$ and $p\in X_k$; we show that $N_\gamma(p)\cap X_k$ is contractible in $N_\varepsilon(p)\cap X_k$. For convenience, let $U=N_\gamma(p)$, and $X_k^m=\{x\in X_k : x_m=0\}$ and $\hat{X}_k^m=\{x\in X_k : x_m=1\}$. Either $U\cap X_k^m=\varnothing$ or $U\cap \hat{X}_k^m=\varnothing$. If $U\cap X_k^m\neq\varnothing$, then $|p_m-0|<\frac{1}{2}$ because, for some $q\in X_k^m$,

$$|p_m - 0|\cdot 2^{-m} \leqq d(p, q) \leqq \gamma < 2^{-(m+1)}.$$

Similarly, if $U\cap\hat{X}_k^m\neq\varnothing$, then $|p_m-1|<\frac{1}{2}$. Hence U cannot intersect both X_k^m and $\hat{X}_k^m$, for that would imply

$$|0 - 1| \leqq |0 - p_m| + |p_m - 1| < \tfrac{1}{2} + \tfrac{1}{2} = 1,$$

a contradiction.

Consider the case $U\cap\hat{X}_k^m=\varnothing$; the proof in case $U\cap X_k^m=\varnothing$ is entirely analogous. For each $q\in U\cap X_k$, let $\tilde{q}\in X_k$ be such that $\tilde{q}(i)=q(i)$ if $i\neq m$, and $\tilde{q}(m)=0$. Now consider the homotopy $H:(U\cap X_k)\times[0,1]\to N_\varepsilon(p)\cap X_k$ defined by moving each point q along three line segments, first to $\tilde{q}$, then to $\tilde{p}$, and finally to p. The mapping H is clearly continuous. Also the distance to p of any point in the image of H is less than ε, because $d(q,\tilde{q})=|q_m-0|\cdot 2^{-m}<\frac{1}{3}\varepsilon$, $d(\tilde{q},\tilde{p})\leqq d(q,p)<\gamma<2^{-m}<\frac{1}{3}\varepsilon$, and $d(\tilde{p},p)=|0-p_m|\cdot 2^{-m}<\frac{1}{3}\varepsilon$. Therefore the distance to p of any point in the image of H is less than ε. Finally, if $q\in U\cap X_k$ then at least one of three alternatives holds:

(1) $q_1\in\{(k+1)^{-1}, k^{-1}\}$,
(2) $q_i\in\{0,1\}$ for $i=2,\cdots,m-1,m+1,\cdots,k+1$, or
(3) $q_m=0$.

One notes that each point of the line segment from $\tilde{q}$ to q also satisfies one of the three conditions and therefore belongs to X_k. Also $\tilde{q}(m)=\tilde{p}(m)=0$, so that the line segment from $\tilde{q}$ to $\tilde{p}$ lies in X_k. Therefore, the image of the homotopy is contained in X_k.

Given $\varepsilon>0$, we have shown how to find m and $\gamma>0$ such that γ illustrates the uniformly equi-*LC* property for all $X_k\in\phi(Q)$ such that $k\geqq m$.

But when $2 \leqq i < m$, there is a number $\delta_i > 0$ illustrating the uniform equi-LC property for the particular i-cell X_i. Let $\delta = \min\{\gamma, \delta_2, \cdots, \delta_{m-1}\}$; then δ illustrates the property for the whole collection $\phi(Q)$. The members of $\phi(Q)$ which are not X_i's are points or Q itself. For points there is nothing to prove; for Q, a linear homotopy (of diameter 2δ) will contract $N_\delta(p)$ to p. The lemma is established, and Theorem 3.1 shows that $\phi(Q)$ is uniformly equi-LAE.

5. **The nonexistence of a selection for ϕ.** In this section we locate a point $p \in Q$ such that there is no selection for ϕ restricted to any neighborhood of p. Let p be the point such that $p_1 = 0$ and $p_i = \frac{1}{2}$ for $i > 1$.

Suppose that U is a neighborhood of p and $f: U \to Q$ is a selection for $\phi | U$. Without loss of generality, we may assume U is of the following form: For some integer $m \geqq 3$ and positive $\varepsilon < 2^{-m}$,

$$U = \{x \in Q : x_1 \leqq m^{-1} \text{ and } |x_i - \tfrac{1}{2}| \leqq \varepsilon \text{ if } i \text{ satisfies } 2 \leqq i \leqq m\}.$$

For each integer $k \geqq 2$, recall the definitions of E_k, X_k and $\tilde{X}_k$ from §2. Notice that, for $k \geqq 2$, if $q \in U \cap \tilde{X}_k$ then $f(q) \in X_k$, so that $f(q)_1 \in \{(k+1)^{-1}, k^{-1}\}$ or $f(q)_i \in \{0, 1\}$ for some i satisfying $2 \leqq i \leqq k+1$. Also if $q \in U \cap X_k$ then $f(q) = q$. For each $k \geqq m$, define

$$Y_k = \{x \in E_k : x_i = p_i \text{ for } 2 \leqq i \leqq m\}.$$

Let $\partial Y_k = Y_k \cap X_k$, and note that $Y_k - \partial Y_k = Y_k \cap \tilde{X}_k$. We see that $x \in \partial Y_k$ implies that $x_1 \in \{(k+1)^{-1}, k^{-1}\}$ or $x_j \in \{0, 1\}$ for some $j = m+1, \cdots, k+1$. Also, $x \in Y_k - \partial Y_k$ implies $x_1 \in ((k+1)^{-1}, k^{-1})$ and $x_j \in (0, 1)$ for $j = m+1, \cdots, k+1$. Therefore Y_k is a closed $(k-m+2)$-cell with (manifold) interior and boundary $Y_k - \partial Y_k$ and ∂Y_k, respectively. Observe that $Y_k \subseteq N$ for each $k \geqq m$.

We claim that, for each $k \geqq m$, there exists $y_k \in Y_k - \partial Y_k$ such that, for some i satisfying $2 \leqq i \leqq m$, $f(y_k)_i \in \{0, 1\}$, and hence $d(y_k, f(y_k)) \geqq 2^{-(m+1)}$. Suppose not; then let

$$r : \{x \in X_k : x_i \in (0, 1) \text{ for } i \text{ satisfying } 2 \leqq i \leqq m\} \to Y_k$$

be the continuous function defined by $r(x)_i = p_i$ if $2 \leqq i \leqq m$, and $r(x)_j = x_j$ if $j = 1$ or $m < j \leqq k+1$. The function $r \circ f | Y_k$ is then a retraction of the $(k-m+2)$-ball Y_k onto its boundary ∂Y_k: In fact, if $y \in Y_k - \partial Y_k \subseteq \tilde{X}_k$, then $f(y) \in X_k$; and if $y \in \partial Y_k \subseteq X_k$, then $f(y) = y$. This is a contradiction.

For each $k \geqq m$, let $y_k \in Y_k - \partial Y_k$ such that $d(y_k, f(y_k)) \geqq 2^{-(m+1)}$. Since U is compact, the sequence $y_m, y_{m+1}, \cdots$ has a subsequence which converges to a point $q \in U$. Since $(y_k)_1 \in [(k+1)^{-1}, k^{-1}]$, we see that $q_1 = 0$, i.e., $q \in X_\omega$ and $f(q) = q$. But $d(y_k, f(y_k)) \geqq 2^{-(m+1)}$ for all $k \geqq m$, so $d(q, f(q)) \geqq 2^{-(m+1)}$. This contradiction completes the proof.

6. **Open problems.** The problem of finding a useful selection theory for carriers $\phi: X \to \mathcal{S} \subseteq 2^Y$, where X is infinite dimensional and the conditions on ϕ, X, $\mathcal{S}$, and Y are "topological," remains unsolved. Selection theory in which the dimension of X is arbitrary, and each $\phi(x)$ satisfies "convexity" conditions, is investigated in [**5**] and [**8**].

Because of similarities between E. Michael's Theorem 1.2 [**6**] and the theory of finite-dimensional *ANE*'s and because of our Theorem 1.1, it is plausible that a condition which characterizes infinite-dimensional *ANE*'s is relevant to the solution of the infinite-dimensional selection problem.

S. Lefschetz [**4**, pp. 84–87, (6.6)] showed that a compact metric space Y is an *ANE* if and only if it satisfies the following condition:

CONDITION (L). For every $\varepsilon > 0$, there exists $\delta > 0$ such that, if

(1) K is any polyhedron,

(2) L is any subpolyhedron of K which contains all the vertices of K, and

(3) $f: L \to Y$ is a continuous function such that, $\operatorname{diam}(f(L \cap \sigma)) < \delta$ for each simplex σ of K,

then there exists a continuous extension $\tilde{f}: K \to Y$ such that $\operatorname{diam}(\tilde{f}(\sigma)) < \varepsilon$ for each simplex σ of K.

We say a collection $\mathcal{S} \subseteq 2^Y$ is *uniformly equi-*(L) if each $S \in \mathcal{S}$ satisfies Condition (L), and if the choice of δ depends on ε but is independent of $S \in \mathcal{S}$. Every uniformly equi-(L) collection $\mathcal{S}$ is uniformly equi-*LAE*, but not conversely. The first is easily seen from Lefschetz [**4**, (6.6)] and Dugundji [**2**, Theorem 3.1],[7] and the second follows from the fact that $\phi(Q)$ in the proof of Theorem 1.1 is not uniformly equi-(L).[8]

Another relevant condition may be the condition that ϕ is continuous. (I.e., if $x_0 \in X$ and $\varepsilon > 0$, then x_0 is interior to the set of points x in X such that the Hausdorff distance between $\phi(x)$ and $\phi(x_0)$ is less than ε. This definition of "continuous" is valid when each $\phi(x)$ is compact.)

We do not know whether either or both of the above conditions, together with the usual conditions of selection theory, will be enough to guarantee a selection for ϕ.

REFERENCES

1. K. Borsuk, *Theory of retracts*, Monografie Mat., Tom 44, PWN, Warsaw, 1967. MR **35** #7306.

[7] Although not explicitly stated there, the construction in [**2**] yields a T_2-normal space.

[8] One observes that for any $\delta > 0$, there is an integer k such that E_k is the body of a simplicial complex whose vertices lie in a subcomplex whose body is X_k. To retract E_k onto X_k would require moving some point distance at least $\frac{1}{2}$.

2. J. Dugundji, *An extension of Tietze's theorem*, Pacific J. Math. **1** (1951), 353–367. MR **13**, 373.

3. O. Hanner, *Retraction and extension of mappings of metric and non-metric spaces*, Ark. Mat. **2** (1952), 315–360. MR **14**, 396; 1278.

4. S. Lefschetz, *Topics in topology*, Ann. of Math. Studies, no. 10, Princeton Univ. Press, Princeton, N.J., 1942 MR **4**, 86.

5. E. Michael, *Continuous selections*. I, Ann. of Math. (2) **63** (1956), 361–382. MR **17**, 990.

6. ———, *Continuous selections*. II, Ann. of Math. (2) **64** (1956), 562–580. MR **18**, 325.

7. ———, *Local properties of topological spaces*, Duke Math. J. **21** (1954), 163–171. MR **15**, 977.

8. ———, *Convex structures and continuous selections*, Canad. J. Math. **11** (1959), 556–575. MR **22** #230.

Department of Mathematics, State University of New York at Binghamton, Binghamton, New York 13901

Department of Mathematics, University of Texas, Austin, Texas 78712