

THE CONDITION $\text{Ext}^i(M, R)=0$ FOR MODULES OVER LOCAL ARTIN ALGEBRAS (R, \mathfrak{M}) WITH $\mathfrak{M}^2=0$

MARGARET S. MENZIN

ABSTRACT. Let M be a finitely generated module over a (not necessarily commutative) local Artin algebra (R, \mathfrak{M}) with $\mathfrak{M}^2=0$. It is known that when R is Gorenstein (i.e. of finite injective dimension) $M=\Sigma R\oplus\Sigma R/\mathfrak{M}$. For R not Gorenstein we describe all M with $\text{Ext}^1(M, R)=0$ and show that $\text{Ext}^i(M, R)=0$ for some $i>1$ if and only if M is free. It follows that for R not Gorenstein all reflexives are free. We also calculate the lengths of all the $\text{Ext}^i(M, R)$. As an application we show that if (R, \mathfrak{M}) is a commutative Cohen-Macaulay local ring of dimension d which is not Gorenstein, if R/\mathfrak{M}^2 is Artin and (x_1, \dots, x_d) is a system of parameters with \mathfrak{M}^2 contained in the ideal (x_1, \dots, x_d) and if M is a finitely generated R -module with $\text{Ext}^i(M, R)=0$ for $1\leq i\leq 2d+2$, then M is free.

We call (R, \mathfrak{M}) a local Artin algebra if R/\mathfrak{M} is a division ring, where \mathfrak{M} is the Jacobson radical of R , if the center of R is an Artin ring, and if R is a finitely generated module over its center. We say R is Gorenstein if it is of finite injective dimension as an R -module. Throughout this paper all modules will be finitely generated.

Let (R, \mathfrak{M}) be a local Artin algebra. It is well known that every finitely generated left R -module M has a projective cover (i.e. an epimorphism $\varphi:P\rightarrow M$ minimal in the sense that $\text{Ker } \varphi\subseteq \mathfrak{M}P$) which is unique up to isomorphism [3], and that M has no projective (free) direct summands if and only if $M^*=\text{Hom}_R(M, R)$ is isomorphic to $\text{Hom}_R(M, \mathfrak{M})$ [2]. Further, for each finitely generated left R -module M there is a minimal presentation—i.e. an exact sequence $F_1\rightarrow F_0\rightarrow M\rightarrow 0$ with $\varphi_0:F_0\rightarrow M$ and $\varphi_1:F_1\rightarrow \text{Ker } \varphi_0$ projective covers, F_0 and F_1 free and finitely generated. We use this (unique) minimal presentation to define the parameters g_M and r_M .

DEFINITION. Let M be a finitely generated left module over a local Artin algebra (R, \mathfrak{M}) . Let

$$\begin{aligned} g_M &= \ell(F_0/\mathfrak{M}F_0) && \text{the number of generators of } M, \\ r_M &= \ell(F_1/\mathfrak{M}F_1) && \text{the number of relations of } M, \end{aligned}$$

where $\ell(\)$ means the length as a left R -module.

Received by the editors July 6, 1972 and, in revised form, June 1, 1973.

AMS (MOS) subject classifications (1970). Primary 16A46, 16A10, 16A64, 13H10; Secondary 16A62.

Key words and phrases. Artin local algebra, reflexive, Cohen-Macaulay, Gorenstein.

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We remark that if (R, \mathfrak{M}) is a local Artin algebra and M is a 2-sided R -module then the lengths of M as a left and as a right R -module coincide.

PROPOSITION 1. *Let (R, \mathfrak{M}) be a local Artin algebra with $\mathfrak{M}^2=0$. Let $n=\ell(\mathfrak{M})\geq 1$ and let M be a finitely generated left R -module with no free direct summands. Then:*

- (a) $\ell(M)=(n+1)g_M-r_M$,
- (b) $\ell(M^*)=ng_M$,
- (c) $\ell(\text{Ext}^1(M, R))=nr_M-g_M$,
- (d) $\ell(\text{Ext}^i(M, R))=n^{i-2}r_M(n^2-1)$, $i\geq 2$.

PROOF. We write g for g_M and r for r_M . Let $0\rightarrow K\rightarrow R^g\rightarrow M\rightarrow 0$ be exact with $R^g\rightarrow M\rightarrow 0$ a projective cover. As $K\subseteq \mathfrak{M}R^g$, $\ell(M/\mathfrak{M}M)=g_M=g$. Also, as $\mathfrak{M}^2=0$, $\mathfrak{M}K=0$ and K is a direct sum of copies of R/\mathfrak{M} . $0\rightarrow \mathfrak{M}\rightarrow R\rightarrow R/\mathfrak{M}\rightarrow 0$ is the projective cover of R/\mathfrak{M} , and \mathfrak{M} is isomorphic to n copies of R/\mathfrak{M} (direct sum). Since $R^r\rightarrow K\rightarrow 0$ is a projective cover, $g_K=\ell(K)=r_M=r$, and by the same reasoning $r_K=\ell(\mathfrak{M}R^r)=nr (=nr_M)$.

- (a) From $\ell(R)=\ell(\mathfrak{M})+1=n+1$, we have

$$\ell(M)=\ell(R^g)-\ell(K)=(n+1)g-r.$$

(b) As M is without free summands and $\mathfrak{M}^2=0$, $M^*\approx \text{Hom}_R(M, \mathfrak{M})\approx \text{Hom}_R(M/\mathfrak{M}M, \mathfrak{M})\approx \text{Hom}_{R/\mathfrak{M}}(M/\mathfrak{M}M, \mathfrak{M})$.

As the lengths of R/\mathfrak{M} modules as R -modules and as R/\mathfrak{M} -modules coincide, this last has length $\ell(M/\mathfrak{M}M)\cdot \ell(\mathfrak{M})=ng$.

(c) From the exact sequence $0\rightarrow M^*\rightarrow (R^g)^*\rightarrow K^*\rightarrow \text{Ext}^1(M, R)\rightarrow 0$, we have

$$\begin{aligned}\ell(\text{Ext}^1(M, R)) &= \ell(M^*) + \ell(K^*) - \ell(R^g)^* \\ &= ng_M + ng_K - (n+1)g_M \\ &= ng_K - g_M = nr_M - g_M = nr - g.\end{aligned}$$

(d) $\text{Ext}^2(M, R)\approx \text{Ext}^1(K, R)$ is of length $nr_K-g_K=n(nr)-r=r(n^2-1)=r_M(n^2-1)$. By induction on i , $\text{Ext}^{i+1}(M, R)=\text{Ext}^i(K, R)$ is of length $n^{i-2}r_K(n^2-1)=n^{i-1}r(n^2-1)=n^{i-1}r_M(n^2-1)$ for $i\geq 2$.

PROPOSITION 2. *Let (R, \mathfrak{M}) be a local Artin algebra with $\mathfrak{M}^2=0$, and let M be a (finitely generated) left R -module. If $n=\ell(\mathfrak{M})\geq 2$ then the following are equivalent:*

- (a) M is free.
- (b) $\text{Ext}^1(M, R)=\text{Ext}^2(M, R)=0$.
- (c) $\text{Ext}^i(M, R)=0$ for some $i\geq 2$.

PROOF. Trivially (a) \Rightarrow (b) \Rightarrow (c). If $M\neq(0)$ is a module without free summands, $n\geq 2$, and some $\text{Ext}^i(M, R)=0$, then by the last formula in Proposition 1, $r_M=0$. Hence $K=0$ and M is free.

We obtain easily the well-known result:

COROLLARY 3. *Let (R, \mathfrak{M}) be a local Artin algebra with $\mathfrak{M}^2=0$. Let $n=\ell(\mathfrak{M})$. Then R is Gorenstein if and only if $n=0$ or 1. If R is Gorenstein, then $\text{Injective dimension}_R R=0$ (i.e., R is injective) and all R -modules are of the form $\Sigma R \oplus \Sigma R/\mathfrak{M}$.*

PROOF. When $n=0$, R is a division ring and the statement is true trivially. If R is Gorenstein, $\text{Ext}^i(R/\mathfrak{M}, R)=0$ for some $i \geq 1$ and also for some $i \geq 2$. Since $r_{R/\mathfrak{M}}=n$, from Proposition 1, we obtain $n^2-1=0$ and $n=1$. If $n=1$, $\text{Ext}^2(M, R)=0$ for all finitely generated M by Proposition 1 and R is of finite injective dimension. In fact, since $r_{R/\mathfrak{M}}=n=1$ and $g_{R/\mathfrak{M}}=1$, we find $\text{Ext}^1(R/\mathfrak{M}, R)=0$ and hence

$$\text{Injective dimension}_R R=0.$$

If R is Gorenstein, and $M \neq R$ is indecomposable, then M is torsion-free (for example, by the sequence $S(\)$ applied to R , below). So $M \subseteq P$ for some P projective and as M is without free summands $M \subseteq \mathfrak{M}P$ [2]. Since $\mathfrak{M}^2=0$, M is not faithful. Hence, \mathfrak{M} is the annihilator of M and $M=R/\mathfrak{M}$. R and R/\mathfrak{M} are the only indecomposable R -modules.

To investigate reflexive modules we introduce the map T . If R is a local Artin algebra and M is a finitely generated left R -module with minimal presentation

$$(1) \quad F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

(F_0, F_1 finitely generated and free), then we define $T(M)=\text{Coker}(F_0^* \rightarrow F_1^*)$ so that

$$(2) \quad 0 \rightarrow M^* \rightarrow F_0^* \rightarrow F_1^* \rightarrow T(M) \rightarrow 0$$

is exact. When M has no free (direct) summands and (1) is a minimal presentation of M , then (2) is a minimal presentation of $T(M)$, and (since $T(M)$ is without free summands) this presentation of $T(M)$ produces $T^2(M)$ with $T^2(M) \approx M$. (To show (2) is a minimal presentation, write down the definitions of the maps. $\pi: F_1^* \rightarrow T(M)$ is minimal when M is without free summands so that $M^*=\text{Hom}_R(M, \mathfrak{M})$, since $\varphi_0: F_0 \rightarrow M$ is minimal. $\varphi_1^*: F_0 \rightarrow \text{Ker } \pi$ is minimal since $\varphi_1: F_1 \rightarrow \text{Ker } \varphi_0$ is minimal.) Also, as minimal presentations of M_1 and M_2 induce a minimal presentation of $M_1 \oplus M_2$, $T(M_1 \oplus M_2) \approx T(M_1) \oplus T(M_2)$.

One of the most important facts about T is that if M is a left R -module then

$$\begin{aligned} S(\) \quad 0 \rightarrow \text{Ext}^1(T(M), _) \rightarrow _ \otimes M \rightarrow \text{Hom}(M^*, _) \\ \rightarrow \text{Ext}^2(T(M), _) \rightarrow 0 \end{aligned}$$

is an exact sequence of functors of right R -modules [1, Proposition 6.3], or [4, p. 43].

In particular, when we examine the sequence $S(R)$ for the right R -module R , the second map is the canonical $M \rightarrow M^{**}$ and we see that M is reflexive ($M \approx M^{**}$) if and only if $\text{Ext}^1(T(M), R) = \text{Ext}^2(T(M), R) = 0$.

PROPOSITION 4. *Let (R, \mathfrak{M}) be a local Artin algebra with $\mathfrak{M}^2 = 0$. If $n = \ell(\mathfrak{M}) \geq 2$ then any reflexive module is free. If $n = \ell(\mathfrak{M}) < 2$ then every module is reflexive.*

PROOF. We use the (unique) minimal presentation of M to define $T(M)$. If $n = \ell(\mathfrak{M}) < 2$, R is Gorenstein, hence all $\text{Ext}^1(T(M), R) = 0 = \text{Ext}^2(T(M), R)$ and every M is reflexive. If $n \geq 2$, let M be a reflexive module without free summands. Then $\text{Ext}^2(T(M), R) = 0$ and by Proposition 2, $T(M)$ is free. But then $0 \rightarrow T(M) \rightarrow T(M) \rightarrow 0$ is the unique minimal presentation of $T(M)$, so $M = T(T(M)) = 0$. Thus the only reflexives are free.

PROPOSITION 5. *Let (R, \mathfrak{M}) be a local Artin algebra with $\mathfrak{M}^2 = 0$. Then the following conditions are equivalent:*

- (a) $\text{Ext}^1(M, R) = 0$.
- (b) M is a direct sum $\Sigma R \oplus \Sigma T(R/\mathfrak{M})$.
- (c) If $0 \rightarrow A \rightarrow B \rightarrow M \rightarrow 0$ and $0 \rightarrow A/\mathfrak{M}A \rightarrow B/\mathfrak{M}B \rightarrow M/\mathfrak{M}M \rightarrow 0$ are both exact, then the first sequence splits.
- (d) If $0 \rightarrow A \rightarrow B \rightarrow M \rightarrow 0$ is exact and minimal presentations of A and M induce a minimal presentation of B , then the sequence splits.

PROOF. (c) and (d) are equivalent since the condition that minimal presentations of A and M induce a minimal presentation of B is exactly the condition that $g_A + g_M = g_B$. Let $n = \ell(\mathfrak{M})$. Now if $M = T(R/\mathfrak{M})$ then $r_M = g_{R/\mathfrak{M}} = 1$ and $g_M = r_{R/\mathfrak{M}} = n$, so $\ell(\text{Ext}^1(M, R)) = nr_M - g_M = 0$. Hence (b) \Rightarrow (a).

To show (a) \Rightarrow (b), from the sequence before Proposition 4 we have the exact sequence $0 \rightarrow \text{Ext}^1(T^2(M), R) \rightarrow T(M) \rightarrow T(M)^{**}$. If $M = F \oplus M'$, M' without free summands, then $T^2(M) = T^2(M') = M'$. By our assumption $0 = \text{Ext}^1(T^2(M), R) = 0$ and $0 \rightarrow T(M) \rightarrow T(M)^{**}$ is exact—i.e. $T(M)$ is torsionless. Thus $T(M) \subseteq F'$ for some free F' . As also $T(M)$ is without projective summands, $T(M) \subseteq F'\mathfrak{M}$. As $\mathfrak{M}^2 = 0$, $T(M)$ is a direct sum of copies of R/\mathfrak{M} and therefore $M' \approx T^2(M) = \Sigma T(R/\mathfrak{M})$.

To show (c) \Rightarrow (a) consider a sequence $0 \rightarrow R \rightarrow N \rightarrow M \rightarrow 0$. Now R is not contained in $\mathfrak{M}N$, else $\mathfrak{M} \subseteq \mathfrak{M}^2N = 0$. As $\ell_R(R/\mathfrak{M}) = 1$, $R/\mathfrak{M} \subseteq N/\mathfrak{M}N$ and $0 \rightarrow R/\mathfrak{M}R \rightarrow N/\mathfrak{M}N \rightarrow M/\mathfrak{M}M \rightarrow 0$ is exact. By our assumption (c) the original sequence is split and $\text{Ext}^1(M, R) = 0$.

Finally we show (a) \Rightarrow (d). If minimal presentations of A and M induce a minimal presentation of B , then dualizing the presentations $G_X \rightarrow F_X \rightarrow X \rightarrow 0$, $X=A, B, M$, we obtain the commutative diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \rightarrow & M^* & \rightarrow & F_M^* & \rightarrow & G_M^* & \rightarrow T(M) \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \rightarrow & B^* & \rightarrow & F_B^* & \rightarrow & G_B^* & \rightarrow T(B) \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \rightarrow & A^* & \rightarrow & F_A^* & \rightarrow & G_A^* & \rightarrow T(A) \rightarrow 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
 \end{array}$$

I II

in which all rows are exact, and the second and third columns are split exact in such a way as to make boxes I and II commutative. By the snake lemma, the sequence $0 \rightarrow M^* \rightarrow B^* \rightarrow A^* \rightarrow T(M) \rightarrow T(B) \rightarrow T(A) \rightarrow 0$ is exact. But $\text{Coker}(B^* \rightarrow A^*) = \text{Ext}^1(M, R) = 0$ by assumption. Hence $0 \rightarrow T(M) \rightarrow T(B) \rightarrow T(A) \rightarrow 0$ is exact, the maps being induced from the G -column. The splitting maps on the G -column induce maps $T(A) \rightarrow T(B)$ and $T(B) \rightarrow T(M)$. It is easy to show that, as the F and G -columns both split, the induced maps on the T -column split it, and hence $T(B) \approx T(M) \oplus T(A)$. Hence $B \approx T^2(B) \approx T(T(M) \oplus T(A)) \approx T^2(M) \oplus T^2(A) \approx M \oplus A$ and the original sequence splits.

COROLLARY 6. *Let (R, \mathfrak{M}) be a local Artin algebra with $\mathfrak{M}^2=0$. If $0 \rightarrow A \rightarrow B \rightarrow M \rightarrow 0$ is exact, $\text{Ext}^1(M, R)=0$, and B is finitely generated without free direct summands, then the sequence splits.*

PROOF. We have $0 \rightarrow A \rightarrow B \rightarrow M \rightarrow 0$ and $0 \rightarrow M^* \rightarrow B^* \rightarrow A^* \rightarrow 0$ exact. If M had free direct summands, so would B . Likewise if A^* had free direct summands so would B^* and hence $B^* = \text{Hom}_R(B, \mathfrak{M})$ (since B has no free summands). Thus the dual sequence gives $ng_B = \ell(B^*) = \ell(M^*) + \ell(A^*) = ng_M + ng_A$, since A, B, M are all without free summands. From the equivalence of (a) and (d) in Proposition 5 and $\text{Ext}^1(M, R)=0$ it follows that the original sequence splits.

As an application of the above, we may generalize Proposition 2 to Cohen-Macaulay rings.

PROPOSITION 7. *Let (R, \mathfrak{M}) be a commutative Cohen-Macaulay local ring of dimension d which is not Gorenstein. Let M be a finitely generated*

left R -module. If (x_1, \dots, x_d) is a system of parameters such that $\mathfrak{M}^2 \subseteq (x_1, \dots, x_d) = \Sigma R x_i$ and R/\mathfrak{M}^2 is Artin, then $\text{Ext}^i(M, R) = 0$ for $1 \leq i \leq 2d + 2$ implies that M is free.

PROOF. When $d=0$, $\mathfrak{M}^2=0$. Since R is not Gorenstein, $n=\ell(\mathfrak{M}) \geq 2$ and the result is just Proposition 2. Assume the result for $0 \leq d \leq k-1$, and we show it for $d=k$. Let $x=x_1$. We consider a projective cover of $M: 0 \rightarrow M_1 \rightarrow F_0 \rightarrow M \rightarrow 0$, and show by the inductive hypothesis that M_1/M_1x is a free R/Rx -module. From $0 \rightarrow R \rightarrow R \rightarrow R/Rx \rightarrow 0$ exact, where the first map is multiplication by x , and the hypotheses on M , we conclude $\text{Ext}_R^i(M_1, R/Rx) = 0$, for $1 \leq i \leq 2k$. As x, x_2, \dots, x_d is a R -regular sequence, x is not a zero-divisor on F_0 , or its submodule M_1 . As x is M_1 -regular there are isomorphisms of R/Rx -modules [4, p. 1]:

$$\begin{aligned} (\dagger) \quad \text{Ext}_R^i(M_1, \quad) &\approx \text{Ext}_{R/Rx}^i(M_1/M_1x, \quad), \\ \text{Ext}_R^i(\quad, R) &\approx \text{Ext}_{R/Rx}^{i+1}(\quad, R/Rx) \quad \text{for } i \geq 1. \end{aligned}$$

Hence $\text{Ext}_{R/Rx}^i(M_1/M_1x, R/Rx) = 0$ for $1 \leq i \leq 2k$. Also, R/Rx is a local Cohen-Macaulay ring of dimension $k-1$ with x_2, \dots, x_d a system of parameters satisfying the hypotheses of the theorem. Applying the second isomorphism of (\dagger) to the R/Rx -module R/\mathfrak{M} , we see that R is Gorenstein (i.e. of finite injective dimension) if and only if R/Rx is. By the inductive hypotheses M_1/M_1x is R/Rx -free, and by the first isomorphism of (\dagger) $0 = \text{Ext}_R^1(M_1, R/\mathfrak{M})$. Thus M_1 is R -free and M is of projective dimension 1 or 0. But for any exact sequence $0 \rightarrow K \rightarrow R^q \rightarrow N \rightarrow 0$ of R -modules, we have $\dots \rightarrow \text{Ext}^1(M, R^q) \rightarrow \text{Ext}^1(M, N) \rightarrow \text{Ext}^2(M, K) = 0$. From $0 = \text{Ext}^1(M, R)$, we have $\text{Ext}^1(M, N) = 0$ for all finitely generated N and M is free.

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DEPARTMENT OF MATHEMATICS, SIMMONS COLLEGE, BOSTON, MASSACHUSETTS 02115