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EXTENSIONS OF MEASURES AND THE VON NEUMANN SELECTION THEOREM[1]

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ABSTRACT. Let (X, B_X) be a Blackwell space, where B_X is the σ-algebra of Borel sets. Then if σ is a finite measure defined on a countably generated sub-σ-algebra $B \subset B_X$, σ can be extended to a Borel measure τ. Equivalently, if X and Y are Blackwell and $f: X \to Y$ is Borel, and μ is a Borel measure carried on $f(X) \subset Y$, then there exists a Borel measure τ on X with $\tau^f = \sigma$, where $\tau^f(E) = \tau(f^{-1}(E))$. We characterize $\{\tau | \tau^f = \sigma\}$ if f is semischlicht.

Let B_X denote the Borel sets of a topological space X. We consider the following measure extension (or equivalently restriction) problem: given a measure (we will always mean finite measure) σ defined on a σ-algebra $B \subset B_X$, can σ be extended to all of B_X, i.e., does there exist a Borel measure τ such that $\tau(E) = \sigma(E)$ for all $E \in B$? It is well known (see [**1**, p. 71], for details) that if B_1 and B_2 are σ-algebras, and B_2 is generated by B_1 and finitely many additional sets, then any measure on B_1 can be extended to B_2. The result is not known for countably generated extensions. We show below (Theorem 5) that if X is a Blackwell space and B is a countably generated sub-σ-algebra of B_X, then any measure on B extends to B_X.

A Blackwell space is a measure space (X, B_X), where X is an analytic subset of a complete separable metric space (c.s.m.). A subset A of a c.s.m. is analytic iff A is the continuous image of a c.s.m. We note that the analytic sets form a proper subset of U_X, the set of absolutely measurable subsets of X, where $E \in U_X$ iff E is $\bar{\mu}$-measurable for all finite Borel measures μ, where "$\bar{\mu}$" denotes the completion of μ, i.e., given μ, there exist $E_1, E_2 \in B_X$ such that $E_1 \subset E \subset E_2$ and $\mu(E_2 - E_1) = 0$. A function g is said to be absolutely measurable if $g^{-1}(V) \in U_X$ for all open V. Details may be found in [**3**], [**4**], or [**5**]. We note that if $X \subset S$, X analytic, S a c.s.m., then $B_X = \{E \cap X | E \in B_S\}$, so elements of B_X are topologically analytic, and not necessarily Borel in S.

We begin by considering a special class of sub-σ-algebras of B_X. Let $f: X \to Y$ be Borel measurable, and let $B_f = \{f^{-1}(E) | E \in B_X\}$. Given a Borel

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measure τ on X, let τ^f be the Borel measure in Y defined by $\tau^f(E)=\tau(f^{-1}(E))$. It follows from a well known isomorphism theorem (see [7, p. 270]) that if τ is a positive Borel measure on a c.s.m. X, and μ is a positive Borel measure on a c.s.m. Y with $\mu(Y)=\tau(X)$ and $\tau(\{x\})=0$ for all $x \in X$, then there exists a Borel function $f:X\to Y$ with $\tau^f=\mu$. The following shows that a dual of this is equivalent to our extension problem.

PROPOSITION 1. *Let $f:X\to Y$ be Borel measurable. Then every measure on B_f is the restriction of a Borel measure iff for every μ defined on B_Y with $\bar{\mu}(f(X)^c)=0$ (i.e., μ is carried on $f(X)$), there exists τ on B_X with $\tau^f=\mu$.*

PROOF. Assume that all Borel μ carried on $f(X)\subset Y$ are of the form $\mu=\tau^f$, and let σ be a measure on $B_f\subset B_X$. Define μ on B_Y by $\mu(E)=\sigma(f^{-1}(E))$, $E\subset Y$. By assumption, there exists a Borel measure τ with $\tau^f=\mu$. Thus, $\tau(f^{-1}(E))=\mu(E)=\sigma(f^{-1}(E))$, so τ extends σ.

Conversely, suppose the extension property holds. Then given a Borel measure μ carried on $f(X)\subset Y$, define σ on B_f by $\sigma(f^{-1}(E))=\mu(E)$. (Note σ is well-defined since μ is carried on $f(X)$.) By assumption, σ is the restriction of some τ defined on B_X, and clearly $\tau^f=\mu$.

The proof of our main result relies on the following "selection" theorem of von Neumann.

THEOREM 2 [**6**]. *Let A be an analytic subset of a c.s.m. S, and let F be a continuous real-valued function on A. Then there exists an absolutely measurable $G:F(A)\to A$ such that $F\circ G$ is the identity on $F(A)$.*

COROLLARY 3. *If X and Y are Blackwell and $F: X\to Y$ is Borel, there exists an absolutely measurable $G: Y\to X$ such that $F\circ G$ is the identity on $F(X)$.*

PROOF. Since Y is Blackwell, we may assume $Y\subseteq \boldsymbol{R}$, the real line. Let $(F\times I):(X\times Y)\to(Y\times Y)$ be defined by $(F\times I)((x,y))=(F(x),y)$, and let P_j be projection on the jth coordinate, $j=1, 2$. Then

$$A = \{(x, F(x)) \mid x\in X\} = (F\times I)^{-1}(\{(y,y)\})$$

is Borel in $X\times Y$, and is thus an analytic subset of a c.s.m. Since $P_2: A\to Y$ is continuous, there exists an absolutely measurable $g:P_2(A)\to A$ with $P_2\circ g$ the identity on $P_2(A)=F(X)$. Then $G=P_1\circ g$ satisfies the theorem.

REMARK 4. (i) If $f:X\to Y$ is Borel and B_Y is countably generated, then B_f is countably generated.

(ii) If X is Blackwell and $B\subset B_X$ is countably generated, then there exists a Borel $f:X\to\boldsymbol{R}$ with $B=B_f$. (This was pointed out to me by the referee.)

PROOF. (i) follows immediately.

Suppose $\{E_n\}$ generates the σ-algebra B. Consider $f(x)=\sum_n 3^{-n}\chi_{E_n(x)}$. Then since B and B_f clearly have the same atoms and X is Blackwell, it follows that $B=B_f$ (see [**5**, p. 38]).

THEOREM 5. *Let X be Blackwell and let σ be a finite measure on a countably generated sub-σ-algebra B. Then σ has an extension τ to the full σ-algebra B_X.*

We first prove the following

LEMMA. *If $f:X\to Y$ is absolutely measurable and $S\subset Y$ is absolutely measurable, then $f^{-1}(S)\subset X$ is absolutely measurable. In particular, a composition of absolutely measurable functions is absolutely measurable.*

PROOF. Given any μ on B_X, consider $\nu=(\bar{\mu})^f$ defined on B_Y. Since S is absolutely measurable, there exist S_1, $S_2\in B_Y$ with $S_1\subset S\subset S_2$, $\nu(S_2-S_1)=0$. Let $T_i=f^{-1}(S_i)$. Then $T_1\subset f^{-1}(S)\subset T_2$, T_1, T_2 are absolutely measurable, and $\bar{\mu}(T_2-T_1)=\bar{\mu}(f^{-1}(S_2-S_1))=\nu(S_2-S_1)=0$. Thus, $f^{-1}(S)$ is $\bar{\mu}$-measurable.

PROOF OF THEOREM 5. Given σ on B, we have $B=B_f$ where $f:X\to\boldsymbol{R}$ is Borel. Then there exists an absolutely measurable $g:f(X)=Y\to X$ with $f\circ g=i_Y$. Define τ on B_X by $\tau(S)=\bar{\sigma}((g\circ f)^{-1}(S))$. (This makes sense, since $g\circ f$ is absolutely measurable.) Clearly, if $S\in B_f$, there exists $T\subset Y$ such that $S=f^{-1}(T)$ and $T\in B_Y$.

Then

$$\begin{aligned}\tau(S) &= \bar{\sigma}((g\circ f)^{-1}(S)) = \bar{\sigma}(f^{-1}(g^{-1}(S)))\\ &= \bar{\sigma}(f^{-1}((f\circ g)^{-1}(T))) = \bar{\sigma}(f^{-1}(T))\\ &= \bar{\sigma}(S) = \sigma(S).\end{aligned}$$

Thus, τ extends σ to B_X.

COROLLARY 6. *Let $f:X\to Y$ be Borel, where X and Y are Blackwell, and let μ be a Borel measure carried on $f(X)\subset Y$. Then there exists τ on B_X with $\tau^f=\mu$.*

The proof of the corollary is explicit in the proof of the theorem. We note that the τ obtained is of the form $\tau(E)=\mu(g^{-1}(E))=\mu(f(E\cap S))$, where $S=g(Y)$. S thus consists of one point "selected" from each preimage set $f^{-1}(\{y\})$. Hence, τ is carried on a 1-1 set, i.e., a section, of f.

If f is 1-1, then τ is clearly unique. Suppose that f is semischlicht, i.e., that $f^{-1}(\{y\})$ is a countable set for all $y\in Y$. Then there is a countable collection of disjoint Borel sets $\{S_n\}$ such that $\bigcup S_n=X$, $f_n=f|_{S_n}$ is 1-1, and $f(S_n)=f(X-\bigcup_{j<n}S_j)$. (See [**2**, p. 335].) Given a Borel measure σ on Y, let τ_n be defined on B_X by $\tau_n(E)=\sigma(f(E\cap S_n))$. Then $\tau_n^{f_n}=\sigma|_{f_n(X)}$.

Choose $a_n \in L^1(\tau_n)$ such that $\sum_n a_n(f_n^{-1}(y))=1$ a.e. $[\sigma]$, where

$$a_n(f_n^{-1}(y)) = 0 \quad \text{if } y \notin f_n(S_n),$$

and the series is absolutely convergent. Let τ be defined on B_X by $\tau(E)=\sum_n \int a_n(x)\, d\tau_n(x)$, i.e., $d\tau(x)=\sum_n a_n(x)\, d\tau_n(x)$ so τ is a "convex combination" of the measures τ_n. Then it is easy to see that $\tau^f=\sigma$, and we see below that this characterizes $\{\tau | \tau^f=\sigma\}$.

PROPOSITION 7. *Let X and Y be Blackwell, and let $f: X\to Y$ be a semischlicht Borel map. Let σ be a measure defined on B_Y, and suppose that $\{S_n\}$, $\{f_n\}$ and $\{\tau_n\}$ are as above. Then there exists $\{a_n(x)\}$ such that $a_n \in L^1(\tau_n)$, $\sum_n a_n(f_n^{-1}(y))=1$ a.e. $[\sigma]$, and $d\tau(x)=\sum_n a_n(x)\, d\tau_n(x)$.*

PROOF. Using the Hahn decomposition, we may assume that σ and τ are positive. Then for $E\subset S_n\subset X$,

$$\tau(E) = \tau(f_n^{-1}(f_n(E))) \leqq \tau(f^{-1}(f_n(E))) = \sigma(f_n(E)) = \tau_n(E),$$

so by the Radon-Nikodym theorem, there exists $a_n \in L^1(\tau_n)$ with

$$\tau(E) = \int_E a_n(x)\, d\tau_n(x) = \int_{f_n(E)} a_n(f_n^{-1}(y))\, d\sigma(y) \quad \text{if } E \subset S_n.$$

Thus, for $E\subset X$,

$$\tau(E) = \sum_n \tau(E \cap S_n) = \int_E \sum a_n(x)\, d\tau_n(x),$$

so

$$d\tau(x) = \sum_n a_n(x)\, d\tau_n(x).$$

For $E\subset Y$,

$$\sigma(E) = \tau(f^{-1}(E)) = \sum \int_{f_n(f^{-1}(E)\cap S_n)} a_n(f_n^{-1}(y))\, d\sigma(y)$$
$$= \int_E \left(\sum_n a_n(f_n^{-1}(y))\right) d\sigma(y).$$

Hence,

$$\sum_n a_n(f_n^{-1}(y)) = 1 \text{ a.e. } [\sigma].$$

We point out that if, in fact, $\tau(E)=\sigma(f(E\cap S))$ for $E\subset X$, where $f|_S$ is 1-1, then for each n, there exists $D_n\subset S_n$ such that $a_n=\chi_{D_n}$ and $S=\bigcup_n D_n$. Further, the following example shows that the assumption f semischlicht cannot be dropped.

For $a\in[0,1]$ an irrational number, consider the unique decimal representation $a=0.a_1a_2\cdots$, and let $f:[0,1]\to[0,1]$ be defined by $f(0.a_1a_2a_3\cdots)=0.a_1a_3a_5\cdots$. Then, after defining f in an appropriate manner on the rational points, which we can ignore since we will be using nonatomic measures, f maps $[0, 1]$ onto $[0, 1]$, and

$$f^{-1}(0.a_1a_2\cdots) = \{0.a_1x_1a_2x_2\cdots \mid x_i \text{ arbitrary}\}.$$

Let $\Phi\colon [0, 1]\to([0, 1]\times[0, 1])$ be defined by

$$\Phi(0.a_1a_2a_3\cdots) = (0.a_1a_3a_5\cdots, 0.a_2a_4a_6\cdots).$$

Let m be Lebesgue measure in $[0, 1]$. Let σ be a singular nonatomic measure on $[0, 1]$. We now define τ on $[0, 1]$ by $\tau(E)=(\sigma\times m)(\Phi(E))$. We note that $\Phi(E)$ is measurable since Φ restricted to the irrationals is a homeomorphism.

Suppose $E=f^{-1}(B)$ for some Borel set $B\subset[0, 1]$. Then

$$E = \{0.a_1x_1a_2x_2\cdots \mid 0.a_1a_2\cdots\in B\} \quad\text{and}\quad \Phi(E) = B\times[0, 1].$$

Thus,

$$\begin{aligned}\tau(E) &= \tau(f^{-1}(B)) = (\sigma\times m)(\Phi(E))\\ &= (\sigma\times m)(B\times[0, 1]) = \sigma(B).\end{aligned}$$

Hence, $\tau^f=\sigma$, and it is easy to see that if $f|_S$ is 1-1, then $\tau(S)=0$, so τ cannot be written as a "combination" of measures supported on 1-1 sets of f.

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