

HARMONIC NULL SETS AND THE PAINLEVÉ THEOREM

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ABSTRACT. A less restrictive condition on an open Riemann surface than has been formerly known for a subset of the ideal boundary of a resolutive compactification to have harmonic measure zero is demonstrated. Then a generalized version of a classical theorem of Painlevé is established in this framework.

Recently, Arsove and Leutwiler [2] proved a generalization of the classical theorem of Painlevé which states that an analytic function on a Jordan region which tends to zero at each point of a nondegenerate boundary arc vanishes identically. Their result stemmed from an important new characterization of harmonic null sets. It is well known [1] that the existence of a *positive* harmonic function on a bounded region which tends to ∞ on a boundary set E is necessary and sufficient for E to have harmonic measure zero. In [2], the requirement of positiveness is disposed of.

A similar situation has existed in the field of potential theory on Riemann surfaces. The existence of a *positive* superharmonic function on R whose \liminf tends to ∞ at each point of a subset of a suitable ideal boundary, has long been accepted as a sufficient condition for the subset to have harmonic measure zero. However, it can be demonstrated that here also *positiveness is not required*, and Painlevé's theorem can be extended to Riemann surfaces as well.

THEOREM. *Let R^* be a resolutive compactification of an open Riemann surface R , and $\Delta = R^* - R$. If a superharmonic function s on R tends to ∞ at all points of $E \subset \Delta$, then the harmonic measure ω of E is zero.*

PROOF. Let $G = \{z \in R \mid s(z) > 0\}$. Then G is open in R , and set

$$A = \left\{ \zeta \in \Delta - \text{cl}(R - G) \mid \liminf_{z \rightarrow \zeta} s(z) = \infty \right\}.$$

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By Hilfssatz 8.8 of [3], $\omega(A)=0$. For any $\zeta \in E$, if $\zeta \notin A$, then we must have $\zeta \in \text{cl}(R-G)$ since $\lim_{z \rightarrow \zeta} s(z) = \infty$. Hence there exists a net $\{z_\alpha\} \subset R-G$ such that $z_\alpha \rightarrow \zeta$. But $\zeta \in E$ implies that $s(z_\alpha) \rightarrow \infty$, which contradicts the fact that $s(z_\alpha) \leq 0$. It follows that $E \subset A$ and $\omega(E)=0$ as desired.

The theorem is also valid in the theory of harmonic spaces.

Painlevé's classical result can now be generalized to the following:

COROLLARY. *Let f be analytic on R , $f \rightarrow 0$ at all points of $E \subset \Delta$, where $\omega(E) > 0$. Then $f \equiv 0$.*

PROOF. If $f \not\equiv 0$, let $u = -\log|f|$. Then u is superharmonic on R , and $u \rightarrow \infty$ at all points of E . Thus $\omega(E)=0$, a contradiction.

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