

A MAXIMUM PRINCIPLE FOR n -METAHARMONIC FUNCTIONS

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ABSTRACT. A class of n -metaharmonic functions is shown to satisfy the inequality, $|u(x)| \leq k|u(x_0)|$, where x is an arbitrary point in a domain \bar{D} , x_0 is some fixed point on the boundary of D , and k is a constant.

1. Consider a differential equation of the form

$$(1) \quad Lu \equiv \Delta^n u + a_{n-1} \Delta^{n-1} u + \cdots + a_0 u = 0$$

where a_0, \dots, a_{n-1} are real constants, Δ is the Laplace operator in Euclidean space of p dimensions E^p and $\Delta^m \equiv \Delta(\Delta^{m-1})$, $\Delta^0 \equiv I$. Points in E^p are denoted by $x = (x_1, \dots, x_p)$. Let D be a bounded domain in E^p with a sufficiently smooth boundary ∂D . A function u of class $C(\bar{D}) \cap C^{2n}(D)$ and satisfying (1) in D is called an n -metaharmonic function or regular solution [1]. Associated with (1) is the following characteristic equation

$$(2) \quad f(\lambda) \equiv \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_0 = 0.$$

The main purpose of this paper is to prove the following maximum principle.

THEOREM 1. *Suppose the roots λ_i of (2) satisfy both of the conditions:*

(a) $\operatorname{Re} \lambda_i \geq 0$.

(b) *If $\operatorname{Re} \lambda_i = 0$, then λ_i is a simple root.*

If u is a regular solution of (1) satisfying

$$(3) \quad \Delta u = \Delta^2 u = \cdots = \Delta^{n-1} u = 0 \quad \text{on } \partial D$$

then there exists a positive constant k and a point $x_0 \in \partial D$ such that

$$(4) \quad |u(x)| \leq k |u(x_0)| \quad \text{for all } x \in \bar{D}.$$

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PROOF. Let u be a regular solution of (1). Then the n functions $u_1=u$, $u_2=\Delta u$, \dots , $u_n=\Delta^{n-1}u$ satisfy the vector equation

$$(5) \quad \Delta U = -AU$$

where

$$U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \quad A = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix}$$

Note that $-A$ is the companion matrix of the polynomial $f(\lambda)$ and consequently the eigenvalues μ_i of A are given by $\mu_i = -\lambda_i$ and satisfy both of the conditions:

(a') $\operatorname{Re} \mu_i \leq 0$.

(b') If $\operatorname{Re} \mu_i = 0$, then μ_i is simple.

According to a well-known result in Liapunov stability theory [2, Theorem 4*, p. 183] there exists a positive definite matrix B and a positive semidefinite matrix C satisfying

$$(6) \quad A^T B + BA = -C$$

where " T " denotes the transpose. Defining

$$(7) \quad V = U^T B U$$

and applying the Laplacian to (7) gives

$$(8) \quad \Delta V = (\Delta U)^T B U + U^T B (\Delta U) + 2 \sum_{i=1}^p (U_{x_i})^T B U_{x_i}.$$

Substituting (5) into (8), using (6), and recalling that B is positive definite and C is positive semidefinite, we find that

$$\Delta V \geq -U^T (A^T B + BA) U = U^T C U \geq 0.$$

According to the maximum principle for subharmonic functions, there exists a point $x_0 \in \partial D$ such that

$$(9) \quad V(x_0) \geq V(x)$$

for all $x \in \bar{D}$. Since B is positive definite it follows that the diagonal elements b_{ii} of B are positive and moreover that there exists a positive constant K (in fact the lowest eigenvalue of B) such that $U^T B U \geq K U^T U$.

In view of these remarks and the fact that u satisfies the boundary conditions (3), the inequality in (9) becomes

$$b_{11}u^2(x_0) \geq KU^TU \geq Ku^2(x)$$

and consequently, $|u(x)| \leq k|u(x_0)|$ for all $x \in \bar{D}$, where $k = (b_{11}/K)^{1/2}$.

REMARKS.

1. If instead of (3) we prescribe the boundary conditions

$$u = \Delta u = \cdots = \Delta^{i-1}u = \Delta^{i+1}u = \cdots = \Delta^{n-1}u = 0$$

where i is a given integer such that $0 < i \leq n-1$, then it can be shown, similarly, that there exist an $x_0 \in \partial D$ and a positive constant k_i such that $|\Delta^i u(x)| \leq k_i |\Delta^i u(x_0)|$ for all $x \in \bar{D}$.

2. As a consequence of the inequality (4), the usual type of uniqueness and continuous dependence theorems follow. However, we shall not pursue this point here.

2. Examples.

1. For $n=2$, consider

$$(10) \quad \begin{aligned} \Delta^2 u + a_1 \Delta u + a_0 u &= 0 \quad \text{in } D, \\ \Delta u &= 0 \quad \text{on } \partial D. \end{aligned}$$

The associated characteristic equation $\lambda^2 + a_1 \lambda + a_0 = 0$ has roots

$$\lambda_1 = (-a_1 + (a_1^2 - 4a_0)^{1/2})/2, \quad \lambda_2 = (-a_1 - (a_1^2 - 4a_0)^{1/2})/2.$$

Hence, the inequality (4) is valid provided one of the following conditions is satisfied:

- (i) $a_1^2 - 4a_0 \leq 0$, $a_1 < 0$,
- (ii) $a_1^2 - 4a_0 > 0$, $a_1 < 0$, $a_0 > 0$,
- (iii) $a_1 = 0$, $a_0 > 0$.

We note that this result was obtained in [3] under condition (iii).

The inequality (4) is easily shown to be false, in general, if $a_1 > 0$. In fact, if we let D denote the region $0 < x < \pi$, $0 < y < \pi$, then the problem

$$\begin{aligned} \Delta^2 u + 4 \Delta u + 4u &= 0 \quad \text{in } D, \\ u &= \Delta u = 0 \quad \text{on } \partial D, \end{aligned}$$

has as a solution $u = \sin x \sin y$, for which (4) does not hold.

2. For arbitrary n , consider

$$\begin{aligned} (\Delta - a)^n u &= 0 \quad \text{in } D, \\ \Delta u &= \Delta^2 u = \cdots = \Delta^{n-1} u = 0 \quad \text{on } \partial D. \end{aligned}$$

The associated characteristic equation has only one distinct root, namely $\lambda = a$. Hence if $a > 0$ then inequality (4) is valid. Moreover, the counterexample used in Example 1 shows that inequality (4) is false if $a < 0$.

3. Consider the following equation

$$(11) \quad \Delta^n u + a_0 u = 0 \quad \text{in } D.$$

Since the n th roots of 1 or -1 , for $n \geq 3$, do not satisfy either of the conditions (a) or (b) in Theorem 1, it appears that the inequality (4) is not valid for solutions of (11). However, this is still an open question.

3. Returning to Example 1, it is easily verified that in case (iii) the quadratic expression (7) takes the form

$$V = a_0 u^2 + (\Delta u)^2.$$

By generalizing this expression slightly we are in a position to obtain a maximum principle for the following equation

$$(12) \quad \Delta^2 u + a_0(x)u = 0 \quad \text{in } D$$

where $a_0(x) > 0$, is of class $C(\bar{D}) \cap C^2(D)$. Specifically we set

$$(13) \quad V = \alpha(x)u^2 + \beta(x)(\Delta u)^2$$

where α and β will be determined later. Applying the Laplacian to (13) and making use of (12) gives

$$(14) \quad \begin{aligned} \Delta V &= u^2 \Delta \alpha + 2\alpha |\text{grad } u|^2 + 4u \text{ grad } u \cdot \text{grad } \alpha \\ &+ (\Delta u)^2 \Delta \beta + 2\beta |\text{grad } \Delta u|^2 + 4 \Delta u \text{ grad } \Delta u \cdot \text{grad } \beta \\ &+ 2u \Delta u (\alpha - \beta a_0). \end{aligned}$$

Choosing

$$(15) \quad \alpha > 0, \quad \beta > 0,$$

and setting

$$(16) \quad \alpha = \beta a_0$$

in (14), and simplifying, we obtain

$$\begin{aligned} \Delta V &\geq u^2 [\Delta \alpha - (2/\alpha) |\text{grad } \alpha|^2] + (\Delta u)^2 [\Delta \beta - (2/\beta) |\text{grad } \beta|^2] \\ &+ 2\alpha |\text{grad } u + (u/\alpha) \text{grad } \alpha|^2 + 2\beta |\text{grad } \Delta u + (\Delta u/\beta) \text{grad } \beta|^2 \\ &\geq u^2 [-\alpha^2 \Delta(1/\alpha)] + (\Delta u)^2 [-\beta^2 \Delta(1/\beta)]. \end{aligned}$$

Consequently, V will be subharmonic if in addition to (15) and (16), $1/\alpha$ and $1/\beta$ are superharmonic, i.e.,

$$(17) \quad \Delta(1/\alpha) \leq 0, \quad \Delta(1/\beta) \leq 0.$$

Choosing $\alpha=1$, $\beta=1/a_0$, it follows that (15), (16) and (17) are satisfied provided a_0 is superharmonic, i.e., $\Delta a_0 \leq 0$. Proceeding as in Theorem 1, we find that if $\Delta u=0$ on ∂D , then there exists a point $x_0 \in \partial D$ such that $|u(x)| \leq |u(x_0)|$ for all $x \in \bar{D}$.

Choosing $\alpha=a_0$, $\beta=1$, then (15), (16) and (17) are satisfied provided $1/a_0$ is superharmonic, i.e., $\Delta(1/a_0) \leq 0$. Consequently, if $\Delta u=0$ on ∂D , we find

$$|u(x)| \leq (a_0(x_0)/a_0(x))^{1/2} |u(x_0)| \leq k^* |u(x_0)|$$

where $k^* = (a_0(x_0))^{1/2} / \min_{\bar{D}} (a_0(x))^{1/2}$.

In summary, we have proved the following:

THEOREM 2. *If $a_0(x) > 0$ belongs to class $C(\bar{D}) \cap C^2(D)$ and if*

$$\Delta(1/a_0) \leq 0 \quad \text{or} \quad \Delta a_0 \leq 0,$$

then every regular solution of (12), which satisfies $\Delta u=0$ on ∂D , satisfies inequality (4).

4. The preceding results carry over with little difficulty to the case in which the Laplacian is replaced throughout by the general elliptic operator

$$\sum_{i,j=1}^p a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad a_{ij} = a_{ji},$$

provided one of the two matrices (a_{ij}) or B is diagonal.

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