A SUFFICIENT CONDITION FOR NONVANISHING OF DETERMINANTS

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ABSTRACT. In this note we derive sufficient conditions for a diagonally dominant reducible matrix to be nonsingular.

1. Throughout this note we are concerned with $A=(a_{ij})$, an $n \times n$ matrix which is diagonally dominant and where

$$(1.1) \quad J = \left\{ i \in N \mid |a_{ii}| > \sum_{j=1; j \neq i}^n |a_{ij}| \right\} \neq \emptyset$$

where $N=\{1, 2, \dots, n\}$. If $J=N$, A is strictly diagonally dominant and then the Gersgorin circle theorem implies that the determinant of A does not vanish [1, p. 106]. If A is irreducible, Taussky [5] has shown that A is nonsingular. In this note, we prove the following theorem.

THEOREM. *Let the matrix A be such that for each $i \notin J$ there is a sequence of nonzero elements of A of the form $a_{ii_1}, a_{i_1i_2}, \dots, a_{i_rj}$ with $j \in J$. Then A is nonsingular.*

2. We need the following lemma and results to prove the Theorem.

LEMMA 1. *Let A satisfy the conditions of the theorem. Then for any nonempty subset L of N such that $L \cap J = \emptyset$, there is a nonzero element a_{ij} with $i \in L$ and $j \notin L$.*

PROOF. Let L be a nonempty subset of N such that $L \cap J = \emptyset$. Choose $i_1 \in L$, then $i_1 \notin J$ and, hence, there is a sequence of nonzero elements of A of the form $a_{i_1i_2}, a_{i_2i_3}, \dots, a_{i_{r-1}i_r}$ for some $i_r \in J$. Let r be the first integer such that $i_r \notin L$ and note that $2 \leq r \leq n$ since $i_1 \in L$ and $i_r \notin L$. Then $a_{i_{r-1}i_r} \neq 0$ with $i_{r-1} \in L$ and $i_r \notin L$. This proves Lemma 1.

COROLLARY 2. *Let A satisfy the conditions of the Theorem. If $J=\{i_1, i_2, \dots, i_k\}$, then there is a permutation $(i_1 i_2 \dots i_n)$ of N such that, for each $j=k+1, \dots, n$, $a_{i_ji_l} \neq 0$ for some $l < j$.*

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PROOF. If $J=N$, then there is nothing to prove. Suppose $J \neq N$, then $L_1=N-J$ is nonempty and $L_1 \cap J = \emptyset$. Hence, by Lemma 1, there are an $i_{k+1} \in L_1$ and $j \notin L_1$ such that $a_{i_{k+1}j} \neq 0$. Since $j \notin L_1$, we have $j \in N-L_1=J$. Hence $j=i_l$ for some $1 \leq l \leq k$.

Let $L_2=L_1-\{i_{k+1}\}$. If $L_2=\emptyset$, then the proof is completed. Suppose $L_2 \neq \emptyset$. Obviously $L_2 \cap J = \emptyset$. Again, by Lemma 1, there are $i_{k+2} \in L_2$ and $j \notin L_2$ such that $a_{i_{k+2}j} \neq 0$. Since $j \notin L_2$, we have $j \in J \cup \{i_{k+1}\}$. Hence $j=i_l$ for some $1 \leq l \leq k+1$. The corollary is proved by repeating the above process until $L_p=L_{p-1}-\{i_{k+p-1}\}=\emptyset$.

COROLLARY 3. Let A satisfy the conditions of the Theorem. If $J=\{i_1, i_2, \dots, i_k\}$, then there is a permutation $(i_1 i_2 \dots i_n)$ of N such that

$$(2.1) \quad g_j = |a_{i_j i_j}| - \sum_{l=j+1}^n |a_{i_j i_l}|, \quad j = 1, 2, \dots, n,$$

are positive.

PROOF. From Corollary 2, there is a permutation $(i_1 i_2 \dots i_n)$ of N such that, for each $j=k+1, \dots, n$, $a_{i_j i_l} \neq 0$ for some $l < j$. Notice that if $i_j \in J$, then

$$g_j = |a_{i_j i_j}| - \sum_{l=j+1}^n |a_{i_j i_l}| \geq |a_{i_j i_j}| - \sum_{l=1; l \neq j}^n |a_{i_j i_l}| > 0.$$

Hence $g_j > 0$, $j=1, 2, \dots, k$. For $j=k+1, \dots, n$,

$$\begin{aligned} g_j &= |a_{i_j i_j}| - \sum_{l=j+1}^n |a_{i_j i_l}| \\ &> |a_{i_j i_j}| - \sum_{l=j+1}^n |a_{i_j i_l}| - \sum_{l=1}^{j-1} |a_{i_j i_l}| \geq 0 \end{aligned}$$

since $a_{i_j i_l} \neq 0$ for some $l < j$. This completes the proof.

COROLLARY 4. Let A satisfy the conditions of the Theorem. If A is real and $a_{ij} \leq 0$, $a_{jj} > 0$, then A is an M -matrix [2].

PROOF. Let $J=\{i_1, i_2, \dots, i_k\}$. Then it follows from Corollary 3 that there is a permutation $(i_1 i_2 \dots i_n)$ of N such that

$$g_j = |a_{i_j i_j}| - \sum_{l=j+1}^n |a_{i_j i_l}| > 0, \quad j = 1, 2, \dots, n.$$

Now, $\det A = g_1 \det A(i_1) + \det A\{i_1\}$, where $A(i, j, \dots, k)$ denotes the matrix obtained from A with $i, j, \dots, k$ -rows and columns deleted, and $A\{i_j\}$ is the matrix A with the entry $a_{i_j i_j}$ replaced by $\sum_{l=j+1}^n |a_{i_j i_l}|$. From [1, p. 294, Problem 9], we have $\det A\{i_1\} \geq 0$. Thus

$$\det A \geq g_1 \det A(i_1).$$

Similarly, we have

$$\det A(i_1) \geq g_2 \det A(i_1, i_2),$$

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$$\det A(i_1, i_2, \dots, i_{n-1}) \geq g_n.$$

Hence $\det A \geq \prod_{j=1}^n g_j > 0$.

In the same way, we can show that the determinant of each of the principal submatrix of A is positive, therefore A is an M -matrix.

3. Proof of the Theorem. Ky Fan [2] has shown that if a complex matrix $A = (a_{ij})$ and an M -matrix $B = (b_{ij})$ satisfy

$$(3.1) \quad b_{ii} \leq |a_{ii}| \quad \text{for all } i \in N$$

and

$$(3.2) \quad |a_{ij}| \leq |b_{ij}| \quad \text{for } i \neq j,$$

then $|\det A| \geq \det B$. Our Theorem follows immediately, since the real matrix $B = (b_{ij})$, given by

$$\begin{aligned} b_{ij} &= |a_{ij}| & \text{if } i = j, \\ &= -|a_{ij}| & \text{if } i \neq j, \end{aligned}$$

is an M -matrix (by Corollary 4), and in addition to (3.1) and (3.2), it satisfies

$$(3.3) \quad \det B \geq \prod_{j=1}^n g_j > 0,$$

where the g_j are given in (2.1).

4. For the case where $J = N$, Ostrowski ([3], [4]) has given the following lower bounds for $|\det A|$:

$$(4.1) \quad M_1 = \prod_{i=1}^n \left(|a_{ii}| - \sum_{j=i+1}^n |a_{ij}| \right),$$

and

$$(4.2) \quad M_2 = \prod_{i=1}^n \left(|a_{ii}| - \sum_{j=1}^{i-1} |a_{ij}| \right).$$

Observe that, for $J = N$, the g_j given in (2.1) are positive for any arbitrary permutation $(i_1 i_2 \dots i_n)$ of N . If we choose the permutation $(1 2 \dots n)$, then our lower bound $\prod_{i=1}^n g_i = M_1$. If the permutation is chosen to be $(n (n-1) \dots 1)$, then $\prod_{i=1}^n g_i = M_2$. The following simple example shows that we have a better lower bound.

EXAMPLE 1. Let

$$A = \begin{pmatrix} 3 & 2 & \frac{1}{2} \\ 2 & 5 & 2 \\ \frac{3}{2} & 2 & 4 \end{pmatrix}$$

Then $M_1 = \frac{1}{2} \cdot 3 \cdot 4 = 6$, $M_2 = 3 \cdot 3 \cdot \frac{1}{2} = \frac{9}{2}$ and choosing the permutation (2 3 1), we get

$$\prod_{i=1}^3 g_i = 1 \cdot \frac{5}{2} \cdot 3 = \frac{15}{2}.$$

In general, since for each possible permutation $(i_1 i_2 \cdots i_n)$ of N (obtained from Corollary 3) we can find a corresponding set of g_i 's, and thus we may have more than one lower bound for $|\det A|$.

EXAMPLE 2. Consider the diagonally dominant matrix

$$A = \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 0 \\ 0 & 3 & 3 & 9 & 3 \\ 0 & 0 & 4 & 4 & 8 \end{pmatrix}$$

Clearly A is reducible. Now, $J = \{1\}$ and we have the following sequences of nonzero elements of A :

$$\{a_{21}\}, \quad \{a_{31}\}, \quad \{a_{42}, a_{21}\}, \quad \{a_{53}, a_{31}\}.$$

Hence the matrix A satisfies the conditions of the Theorem.

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