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A FINITELY GENERATED RESIDUALLY FINITE GROUP WITH AN UNSOLVABLE WORD PROBLEM

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ABSTRACT. The group described in the title is obtained as a quotient of a center-by-metabelian group constructed by P. Hall.

It is well known that a residually finite group which is finitely presented has a solvable word problem (V. H. Dyson [1], A. W. Mostowski [6]). It is also known (see the last sentence in V. H. Dyson [1]) although perhaps not well enough (see W. Magnus [5, p. 307]) that a residually finite group which is only finitely generated may have an unsolvable word problem. Herein we exhibit such a group which moreover is soluble, indeed it is center-by-metabelian. This is in some sense a minimal soluble example since all finitely generated metabelian groups have a solvable word problem. (This is an easy consequence of the fact that finitely generated free metabelian groups satisfy the maximal condition on normal subgroups (P. Hall [2]) and thus any finitely generated metabelian group can be almost finitely presented in the sense of Mostowski [6] and is residually finite (P. Hall [3]).)

Credit must be given to Peter Neumann who pointed out the appropriate example to be modified.

Our example uses a recursive function f with a nonrecursive range in a manner similar to the way Higman [5] exhibited a finitely presented group with an unsolvable word problem.

Let G be the center-by-metabelian group constructed by P. Hall in ([2, p. 434]). In the notation of P. Hall, G is generated by a and b and defined by

$$[b_i, b_j, b_k] = 1 \qquad \text{where } b_i = a^{-i}ba^i,\ i, j, k = 0, \pm 1, \cdots,$$
$$c_{ij} = c_{i+k, j+k} \quad \text{where } c_{ij} = [b_j, b_i], j > i,\ i, j, k = 0, \pm 1, \cdots.$$

The center C of G is free abelian with basis

$$d_1, d_2, \cdots \quad \text{where } d_r = c_{i, i+r},\ r = 1, 2, \cdots, i = 0, \pm 1, \cdots.$$

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Let N be the subgroup of G generated by $\{d_{f(n)}^{f(n)!} \mid n=0, 1, \cdots\}$. Clearly N is normal in G and a presentation of G/N is obtained from the presentation of G above by adding the relations

$$d_{f(n)}^{f(n)!} = 1, \qquad n = 0, 1, 2, \cdots .$$

Since f is recursive G/N is recursively presented. On the other hand one cannot decide for an arbitrary integer n whether or not $n \in \operatorname{range} f$. So one cannot decide whether or not $d_n^{n!}=1$ in G/N, i.e. G/N has an unsolvable word problem.

It only remains to show that G/N is residually finite. Since C/N is a direct sum of cycles it is clear that $N=\bigcap_{m=1}^{\infty} C^m N$ and so it suffices to show that for any m, $G/C^m N$ is residually finite. However m divides $r!$ for all but a finite number of integers r thus $C^m N/C^m$ is finite and so it suffices to show that G/C^m is residually finite.

To see that G/C^m is residually finite we let R_k be the normal subgroup of G generated by a^{2k+1}. Then $G/R_k C^m$ can be constructed in a manner analogous to the way P. Hall constructed G.

We define B to be the group generated by the elements $\beta_0, \beta_1, \cdots, \beta_{2k}$ subject to the following defining relations.

(i) $$[\beta_i, \beta_j, \beta_k] = 1, \qquad i, j, k = 0, 1, \cdots, 2k.$$

This makes B nilpotent of class 2 and B' free abelian with basis

$$\gamma_{ij} = [\beta_j, \beta_i], \quad j - i \equiv 1, 2, \cdots, k \bmod 2k + 1;$$

(ii) $$\gamma_{ij} = \gamma_{st} \qquad \text{whenever } j - i \equiv t - s \bmod 2k + 1.$$

Now B' has as a basis

$$\delta_r = \gamma_{i,i+r}, \qquad r = 1, 2, \cdots, k, \quad i = 0, 1, \cdots, 2k$$
$$(i + r \text{ is taken mod } 2k + 1).$$

(iii) $$\delta_r^m = 1, \qquad r = 1, 2, \cdots, k.$$

This gives B' exponent m.

Clearly the map $\beta_i \to \beta_{i+1}$ ($i+1$ is taken mod $2k+1$) is an automorphism of B of order $2k+1$. It is easy to see that $G/R_k C^m$ is the extension of B by this automorphism. It may be defined as the group generated by α and $\beta=\beta_0$ with the above defining relations and

$$\alpha^{2k+1} = 1, \qquad \alpha^{-1}\beta_i\alpha = \beta_{i+1}, \qquad i = 0, 1, \cdots, 2k - 1.$$

Now it is clear for any number of reasons that $G/R_k C^m$ is residually finite—for example it is polycyclic. Furthermore if $g \notin C^m$ then g can be written as a product of a power of a and a finite number of b_i's and

d_r's. Thus there is a k large enough so that $g \notin R_k C^m$ and it follows that G/C^m is residually finite.

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