

ON LOCAL SOLVABILITY OF PSEUDO- DIFFERENTIAL EQUATIONS

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ABSTRACT. A sufficient condition for the local solvability of the equation $u_t - A(x, t, D_x)u = f(x, t)$ is proved, where A is a first order pseudo-differential operator with real symbol. This is a special case of the local solvability conjecture of Nirenberg and Treves.

Introduction. Let $P(x, D)$ be a linear partial differential operator of principal type with smooth complex value coefficients. The question of when the equation $Pu = f$ is locally solvable has been settled by Nirenberg and Treves [4] and Beals and Fefferman [1]. Local solvability is equivalent to the condition:

(P) The imaginary part of P does not change signs on the null bi-characteristics of the real part of P .

For a pseudo-differential operator Nirenberg and Treves conjectured that local solvability is equivalent to the condition:

(Ψ) On every null bicharacteristic of $\operatorname{Re} P$, if $\operatorname{Im} P$ is negative at a point it remains nonpositive from then on.

The purpose of this note is to prove the following special case of this conjecture.

THEOREM 1. Let $P = d/dt - A(t, x, D_x)$ for $(t, x) \in \Omega$ where A is a first order pseudo-differential operator with real symbol $a(t, x, \xi)$. Assume that (Ψ) if $a(t_0, x_0, \xi_0) < 0$ then for $t > t_0$, $a(t, x_0, \xi_0) \leq 0$; and if $a(t_0, x_0, \xi_0) = 0$ then $\operatorname{grad}_{x, \xi} a(t_0, x_0, \xi_0) = 0$.

Then P is locally solvable.

Theorem 1 is a simple consequence of the following a priori estimate for the adjoint of P .

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THEOREM 2. Let $L = d/dt - a(t, x, D_x)$ for $(t, x) \in \Omega = I \times U \subset \mathbb{R}^{n+1}$ (I is an interval containing $t=0$) where A is a first order pseudo-differential operator with real symbol $a(t, x, \xi)$. Assume that if $a(t_0, x_0, \xi_0) > 0$ then $a(t, x_0, \xi_0) \geq 0$ for $t > t_0$. Also assume that if $a=0$ that $\text{grad}_{x,\xi} a=0$. Then given $\varepsilon > 0$, there exists a δ such that

$$\|u\|_0 \leq \varepsilon \|Lu\|_0 \quad \text{for all } u \in C_0^\infty((-\delta, \delta) \times U).$$

The proof of Theorem 2 follows the lines of the constant coefficient case (cf. Nirenberg-Treves [5]). To do this first we localize A as in Hörmander [2]. We need to prove an estimate similar to the sharp Gårding's inequality, but when the symbol does not have constant sign. Lax-Nirenberg [3] have observed that the positivity of the symbol is needed in the proof of Gårding's inequality only to establish that $|\text{grad}_\xi a(x, \xi)|^2 \leq C|a(x, \xi)| |\xi|^{-1}$. But this is a consequence of our second assumption about a .

Proof of Theorem 2. We use the notation of Hörmander [2]. In particular $a_\beta^\alpha = (iD_x)^\beta (iD_\xi)^\alpha a(x, \xi)$.

LEMMA 1. If $a(x, \xi) = 0$ implies $\text{grad}_{x,\xi} a(x, \xi) = 0$ then

$$|\text{grad}_\xi a(x, \xi)|^2 (1 + |\xi|) + |\text{grad}_x a(x, \xi)|^2 (1 + |\xi|)^{-1} \leq C |a(x, \xi)|.$$

PROOF. If $f(x) = 0$ implies $f'(x) = 0$ then $|f(x)|$ is a nonnegative C^2 function. Hence $|f'(x)|^2 \leq 2|f(x)| \max |f''|$. Applying this inequality to each variable in question and using the pseudo-homogeneity of a we get the lemma.

Hereafter we make the convention that C is any constant depending only on the symbol of A .

We now introduce Hörmander's partition of unity. Construct non-negative functions $\phi_j(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\sum_{j=1}^\infty \phi_j^2(x) = 1$ and $x, y \in \text{supp } \phi_j$ implies that $|x - y| \leq C$, and the supports overlap a bounded number of times. Set $\psi_j(\xi) = \phi_j(\xi |\xi|^{-1/2})$ also in $C_0^\infty(\mathbb{R}^n)$. The important properties of the ψ_j are that $\sum \psi_j^2 = 1$,

$$(2) \quad \xi, \eta \in \text{supp } \psi_j \text{ implies } |\xi - \eta| < C |\xi|^{1/2} \quad \text{and}$$

$$(3) \quad \sum_{j=1}^\infty |\psi_j(\eta) - \psi_j(\xi)|^2 \leq \frac{C |\xi - \eta|^2}{|\xi|} + \frac{8 |\xi - \eta|}{|\xi|}.$$

(See Hörmander [2, pp. 141-142] for proofs.) Let ξ_j be any point in the support of ψ_j and set $u^{jk}(x) = \phi_k(x |\xi_j|^{1/2}) \psi_j(D)u$ and $\phi_{jk} = \psi_k(x |\xi_j|^{1/2})$. Observe that

$$(4) \quad \sum_{j,k} \|u^{jk}\|_0^2 = \|u\|_0^2.$$

Choose x^{jk} an arbitrary point in the support of ϕ_{jk} .

LEMMA 2. Depending on whether $a(x^{jk}, \xi^j) \geq 0$ or < 0 ,

$$\operatorname{Re}(\psi^{jk}(x)\psi_j(D)Au, \psi^{jk}(x)\psi_j(D)u) \geq R_{jk}(t) \quad \text{or} \quad \leq R_{jk}(t)$$

where $\sum_{j,k} |R_{jk}(t)| \leq C \|u(t)\|_0^2$.

PROOF OF THEOREM 2. Let $Lu=f$. By the first assumption on a , for each j, k there is a t_{jk} such that $a(t, x_{jk}, \xi) \leq 0$ for $t < t_{jk}$ and ≥ 0 for $t > t_{jk}$. Let $u \in C_0^\infty((-\delta, \delta) \times U)$, δ to be determined in a moment. If $a(t, x_{jk}, \xi_j) \leq 0$ we have that,

$$\begin{aligned} \frac{d \|u^{jk}\|^2}{dt} &= 2 \operatorname{Re}(\phi^{jk}(x)\psi_j(D)u_t, u^{jk}) \\ &= 2 \operatorname{Re}(\phi^{jk}\psi_j(D)(Au + f), u^{jk}) \\ &\leq 2R_{jk} + \|\phi^{jk}(x)\psi_j(D)f\|^2 + \|u_{jk}\|^2 \end{aligned}$$

Upon integrating the above inequality from $-\delta$ to t_{jk} we get

$$\|u^{jk}(t)\|^2 \leq \int_{-\delta}^{t_{jk}} 2R_{jk}(t) + \|\phi^{jk}\psi_j f\|^2 + \|u^{jk}\|^2 dt.$$

Similarly when $a(x^{jk}, \xi_j) \geq 0$ we get

$$\|u^{jk}(t)\|^2 \leq \int_{t_{jk}}^{\delta} 2R_{jk}(t) + \|\phi^{jk}\psi_j f\|^2 + \|u^{jk}\|^2 dt.$$

Combining the last two inequalities, summing over j and k and applying Lemma 2 and equation (4) yields that

$$\begin{aligned} \|u\|_0^2 &= \sum_{j,k} \|u^{jk}\|^2 \leq \int_{-\delta}^{\delta} \sum_{j,k} R_{jk} + \|\phi^{jk}\psi_j(D)f\|^2 + \|u^{jk}\|^2 dt \\ &\leq \int_{-\delta}^{\delta} C \|u\|_0^2 + \|f(t)\|^2 dt. \end{aligned}$$

If we choose δ small enough so that $4\delta C < 1$ it follows that

$$\|u(t)\|^2 \leq 2 \int_{-\delta}^{\delta} \|f(t)\|^2 dt$$

and integrating once more that

$$\int \|u(t)\|^2 dt \leq 2\delta \int_{-\delta}^{\delta} \|f(t)\|^2 dt,$$

which proves Theorem 1.

PROOF OF LEMMA 2. First we will show that,

$$(6) \quad \sum_j |(\psi_j(D)Au, \psi_j(D)u) - (A\psi_j(D)u, \psi_j(D)u)| \leq C \|u\|^2.$$

Let $\hat{a}(\eta, \xi)$ be the Fourier transform of $a(x, \xi)$ with respect to x . We may assume that, outside of U , a has compact support in x , so it then follows that $|\hat{a}(\eta, \xi)| \leq C_N |\xi| (1 + |\eta|)^{-N}$ for any integer N . Using Parseval's theorem, the left-hand side of (6) equals

$$\begin{aligned} & \iint \hat{a}(\eta - \xi, \xi) (\phi_j^2(\eta) - \psi_j(\eta) \psi_j(\xi)) \hat{u}(\xi) \hat{u}(-\eta) d\xi d\eta \\ &= \frac{1}{2} \iint \hat{a}(\eta - \xi, \xi) |\psi_j(\eta) - \psi_j(\xi)|^2 \hat{u}(\xi) \hat{u}(-\eta) d\xi d\eta. \end{aligned}$$

Taking absolute values, summing up over j and applying inequality (3) we may bound the left-hand side of (6) by

$$\iint \frac{C_N |\xi|}{(1 + |\eta - \xi|)^N} \left[\frac{C |\eta - \xi|^2}{|\xi|} + 8 \frac{|\xi - \eta|}{|\xi|} \right] |\hat{u}(\xi)| |\hat{u}(\eta)| d\eta d\xi \leq C \|u\|_0^2$$

if $N \geq n + 1$.

Next setting $A_j = a(x, \xi_j) + \sum_{v=1}^n a^v(x, \xi_j)(D_v - \xi_j)$, we have that

$$(7) \quad |((A - A_j) \psi_j(D)u, \psi_j(D)u)| \leq C \|\psi_j(D)u\|_0^2.$$

Parseval's theorem tells us that the left side of (7) equals

$$\iint \left\{ \hat{a}(\eta - \xi, \xi) - \hat{a}(\eta - \xi, \xi_j) - \sum_v \hat{a}^v(\eta - \xi, \xi_j)(\xi_v - \xi_v^j) \right\} \cdot \psi_j(\eta) \psi_j(\xi) u(-\eta) u(\xi) d\eta d\xi.$$

But $|\xi - \xi_j| \leq C |\xi_j|^{1/2}$ when $\psi_j(\xi) \neq 0$.

Thus by Taylor's theorem the expression in parentheses is

$$O\left(\frac{|\xi_j|^{-1}}{1 + |\eta - \xi|^N} |\xi - \xi_j|^2\right) = O((1 + |\eta - \xi|)^{-N})$$

and the bound follows. Note that

$$\begin{aligned} (8) \quad \operatorname{Re}(\phi(x) A_j u, \phi u) &= \operatorname{Re}\left\{(A_j \phi u, \phi u) + i \int_v a^v \phi(i D_v) \phi |u|^2\right\} \\ &= \operatorname{Re}(A_j \phi u, \phi u) \quad \text{since } a \text{ is real.} \end{aligned}$$

Combining (6), (7) and (8) we have that

$$(9) \quad \sum_{j,k} |\operatorname{Re}(\phi_{jk}(x) \psi_j(D) A u, u^{jk}) - \operatorname{Re}(A_j u^{jk}, u^{jk})| \leq C \|u\|_0^2.$$

Let

$$u^{jk}(x) = \exp\{i\langle x, \xi_j \rangle\} v^{jk}((x - x^{jk}) |\xi_j|^{1/2})$$

and

$$\psi_j(D)u = \exp\{i\langle x, \xi_j \rangle\} v^j(x |\xi_j|^{1/2}).$$

Then $\phi_k(x)v^j$ and v^{jk} differ only by a translation; $|y| \leq C$ if $y \in \text{supp } v^{jk}$ and $|\xi| \leq C$ if $\xi \in \text{supp } v^j$. Therefore it follows that

$$\begin{aligned}
 \int \sum_{k: |\alpha+\beta| \leq N} |y^\beta D^\alpha v^{jk}|^2 dy &\leq C \sum_{k: |\alpha| \leq N} \int |D^\alpha v^{jk}|^2 dy \\
 (10) \qquad \qquad \qquad &\leq C \sum_{k: |\alpha| \leq N} \int |D^\alpha \phi_k(x)v^j(y)| dy \\
 &\leq C \sum_{|\alpha| \leq N} \int |D^\alpha v^j(y)| dy \\
 &\leq C_N |\xi_j|^{n/2} \|\psi_i(D)u\|_0^2.
 \end{aligned}$$

By a change of variables we see that

$$\begin{aligned}
 (A_j u^{jk}, u^{jk}) &= |\xi_j|^{-n/2} \int v^{jk}(y) \left\{ a(x^{jk} + y |\xi_j|^{-1/2}, \xi_j) \right. \\
 &\quad \left. + \sum_v a^v(x^{jk} + y |\xi_j|^{-1/2}, \xi_j) |\xi_j|^{1/2} D_v \right\} v^{jk}(y) dy.
 \end{aligned}$$

By Taylor's theorem, the fact that $\text{supp } v^{jk}$ is bounded, and (10), we have that

$$\begin{aligned}
 \sum_k (A_j u^{jk}, u^{jk}) - |\xi_j|^{-n/2} \int v^{jk}(y) \left\{ a(x^{jk}, \xi_j) + \sum_v a_v(x^{jk}, \xi_j) |\xi_j|^{-1/2} y \right. \\
 (11) \qquad \qquad \qquad \left. + \sum_v a^v(x^{jk}, \xi_j) |\xi_j|^{1/2} D_v \right\} v^{jk}(y) dy \\
 \leq |\xi_j|^{-n/2} \sum_{k, |\beta| \leq 2; |\alpha| \leq 1} \int |y^\beta D^\alpha v^{jk}| dy \leq C \|\psi_j(D)u\|_0^2.
 \end{aligned}$$

Finally we have, by using Lemma 1 and the Cauchy-Schwartz inequality and the bound on the support of v^{jk} ,

$$\begin{aligned}
 \sum_v \int a^v(x^{jk}, \xi^j) |\xi^j|^{1/2} (D_v^* v^{jk}(y)) v^{jk}(y) dy \\
 = \sum_v \int \hat{a}^v(x^{jk}, \xi^j) |\xi^j|^{1/2} \xi_v |\hat{v}^{jk}(\xi)|^2 d\xi \\
 \leq \frac{1}{4} \int |a(x^{jk}, \xi^j)| |\hat{v}^{jk}(\xi)|^2 d\xi + \int \frac{C}{|\xi_j|} |\xi^j| |\xi|^2 |\hat{v}^{jk}(\xi)|^2 d\xi \\
 \leq \frac{1}{4} \int |a(x^{jk}, \xi_j)| |v(y)|^2 dy + |\xi_j|^{n/2} \|u^{jk}\|_0^2.
 \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_v a_v(x^{jk}, \xi^j) |\xi_j|^{-1/2} y |v^{jk}(y)|^2 dy \\ \leq \frac{1}{4} \int |a(x^{jk}, \xi^j)| |v^{jk}(y)|^2 + \int C y^2 |v^{jk}(y)|^2 dy \\ \leq \frac{1}{4} |a(x^{jk}, \xi^j)| |v^{jk}(y)|^2 dy + |\xi_j|^{n/2} \|u^{jk}\|_0^2. \end{aligned}$$

If $a(x^{jk}, \xi^j) \geq 0$ we may combine this with (9) and (11) to get

$$(\phi^{jk}(x) \psi_j(D) Au, u^{jk}) \geq -R_{jk}$$

and $\sum |R_{jk}| \leq C|u|_0^2$. This and similar considerations for the case $a(x^{jk}, \xi^j) < 0$ completes the proof of the lemma.

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