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A RELATIONSHIP BETWEEN CHARACTERISTIC VALUES AND VECTORS

E. T. BEASLEY, JR. AND P. M. GIBSON

ABSTRACT. It is shown that for all nonzero n-component column vectors α and β over a field F there exists a set Γ of n-square matrices over F of cardinality n^2-2n+2 such that, for each n-square matrix A over F, $A\alpha=\alpha$ or $A^T\beta=\beta$ if and only if 1 is a characteristic value of PA for every $P\in\Gamma$.

Let F be a field with unit 1, and A an n-square matrix over F. If all row sums of A or all row sums of A^T (the transpose of A) are 1, then A is a *stochastic* matrix. It is obvious that if A is stochastic then 1 is a characteristic value of PA for every n-square permutation matrix P. R. A. Brualdi and H. W. Wielandt [1] proved the converse. In this paper, it is shown as a corollary to a more general result that the $n!$ permutation matrices can be replaced by a set of n-square matrices of cardinality n^2-2n+2.

Let F^n be the set of all $n\times 1$ column vectors over F, and let $\alpha,\beta\in F^n-\{0\}$. A set Γ of n-square matrices over F is called an (α,β)-*set* provided that, for each n-square matrix A over F, $A\alpha=\alpha$ or $A^T\beta=\beta$ if and only if 1 is a characteristic value of PA for every $P\in\Gamma$. An (α,β)-set Γ is *minimal* if no proper subset of Γ is an (α,β)-set. If $\alpha^T\beta=0$, then α and β are *orthogonal*. Let ε_k be the vector in F^n with all components zero, except component k, which is equal to 1. Denote the vector in F^n with all components equal to 1 by ε. The characterization of stochastic matrices by Brualdi and Wielandt implies that the set of all n-square permutation matrices is an $(\varepsilon,\varepsilon)$-set of cardinality $n!$. We shall show that minimal (α,β)-sets of cardinality n^2-2n+2 can be easily constructed for all $\alpha,\beta\in F^n-\{0\}$.

We first exhibit minimal (α,β)-sets for two particular choices of α and β. One of these is $\alpha=\beta=\varepsilon_1$, and the other is $\alpha=\varepsilon_1$ and $\beta=\varepsilon_n$.

LEMMA 1. *For each positive integer n, the set*

$$\Phi_n=\{\varepsilon_1\varepsilon_1^T\}\cup\{\varepsilon_1\varepsilon_1^T+\varepsilon_i\varepsilon_j^T \mid i,j=2,3,\cdots,n\}$$

of n-square matrices over F is a minimal $(\varepsilon_1,\varepsilon_1)$-set.

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PROOF. Let $A=(a_{ij})$ be an n-square matrix over F. Since $\varepsilon_1^T\varepsilon_1=1$ and $\varepsilon_j^T\varepsilon_1=0$ for $j=2, 3, \cdots, n$,

$$A\varepsilon_1 = \varepsilon_1 \Rightarrow PA\varepsilon_1 = \varepsilon_1 \qquad \forall P \in \Phi_n,$$
$$A^T\varepsilon_1 = \varepsilon_1 \Rightarrow (PA)^T\varepsilon_1 = \varepsilon_1 \quad \forall P \in \Phi_n.$$

Hence, if $A\varepsilon_1=\varepsilon_1$ or $A^T\varepsilon_1=\varepsilon_1$, then 1 is a characteristic value of PA for every $P \in \Phi_n$. Now suppose that 1 is a characteristic value of PA for every $P \in \Phi_n$. Since 1 is a characteristic value of $\varepsilon_1\varepsilon_1^TA$, $a_{11}=1$. Since 1 is a characteristic value of $(\varepsilon_1\varepsilon_1^T+\varepsilon_j\varepsilon_i^T)A$,

$$\det((\varepsilon_1\varepsilon_1^T + \varepsilon_j\varepsilon_i^T)A - I) = 0, \qquad i, j = 2, 3, \cdots, n.$$

This implies that

$$\det\begin{bmatrix} a_{11}-1 & a_{1j} \\ a_{i1} & a_{ij}-1 \end{bmatrix} = 0, \qquad i, j = 2, 3, \cdots, n. \tag{1}$$

Therefore, since $a_{11}=1$, $a_{i1}a_{1j}=0$, $i, j=2, 3, \cdots, n$. Since $a_{11}=1$, this implies that $A\varepsilon_1=\varepsilon_1$ or $A^T\varepsilon_1=\varepsilon_1$. Hence, Φ_n is an $(\varepsilon_1, \varepsilon_1)$-set. We now show that Φ_n is minimal. If

$$A = \varepsilon\varepsilon_1^T + \varepsilon_1\varepsilon^T - 2\varepsilon_1\varepsilon_1^T,$$

then $A\varepsilon_1=A^T\varepsilon_1\neq\varepsilon_1$, and equation (1) holds. This implies that $\Phi_n-\{\varepsilon_1\varepsilon_1^T\}$ is not an $(\varepsilon_1, \varepsilon_1)$-set. If $k, m \in \{2, 3, \cdots, n\}$ and

$$A = \varepsilon_1\varepsilon_1^T + \varepsilon_1\varepsilon_k^T + \varepsilon_m\varepsilon_1^T,$$

then $A\varepsilon_1\neq\varepsilon_1\neq A^T\varepsilon_1$, $a_{11}=1$, and equation (1) holds except for $i=m$ and $j=k$. This implies that

$$\Phi_n - \{\varepsilon_1\varepsilon_1^T + \varepsilon_k\varepsilon_m^T\}, \qquad k, m = 2, 3, \cdots, n,$$

is not an $(\varepsilon_1, \varepsilon_1)$-set. Therefore, Φ_n is a minimal $(\varepsilon_1, \varepsilon_1)$-set.

LEMMA 2. *Let $E_{1n}=\varepsilon_1\varepsilon_1^T+\varepsilon_n\varepsilon_n^T$. For each integer $n>1$, the set*

$$\Psi_n = \{E_{1n}, E_{1n} + \varepsilon_1\varepsilon_n^T\}$$
$$\cup \{E_{1n} + \varepsilon_1\varepsilon_j^T, E_{1n} + \varepsilon_i\varepsilon_n^T, E_{1n} + \varepsilon_1\varepsilon_j^T + \varepsilon_i\varepsilon_n^T \mid i, j = 2, 3, \cdots, n-1\}$$

of n-square matrices over F is a minimal $(\varepsilon_1, \varepsilon_n)$-set.

PROOF. Let $A=(a_{ij})$ be an n-square matrix over F. Then

$$A\varepsilon_1 = \varepsilon_1 \Rightarrow E_{1n}A\varepsilon_1 = \varepsilon_1, \varepsilon_i\varepsilon_{j+1}^TA\varepsilon_1 = 0, \qquad i, j = 1, 2, \cdots, n-1,$$
$$A^T\varepsilon_n = \varepsilon_n \Rightarrow (E_{1n}A)^T\varepsilon_n = \varepsilon_n, (\varepsilon_i\varepsilon_{j+1}^TA)^T\varepsilon_n = 0, \qquad i, j = 1, 2, \cdots, n-1.$$

Hence, if $A\varepsilon_1=\varepsilon_1$ or $A^T\varepsilon_n=\varepsilon_n$, then 1 is a characteristic value of PA for every $P\in\Psi_n$. Suppose that 1 is a characteristic value of PA for every $P\in\Psi_n$. Since 1 is a characteristic value of $E_{1n}A$ and $(E_{1n}+\varepsilon_1\varepsilon_n^T)A$,

$$\det\begin{bmatrix} a_{11}-1 & a_{1n} \\ a_{n1} & a_{nn}-1 \end{bmatrix}=0, \tag{2}$$

$$\det\begin{bmatrix} a_{11}+a_{n1}-1 & a_{1n}+a_{nn} \\ a_{n1} & a_{nn}-1 \end{bmatrix}=0. \tag{3}$$

Equations (2) and (3) imply that $a_{n1}=0$. Then equation (2) implies that $a_{11}=1$ or $a_{nn}=1$. Hence, if $n=2$, then $A\varepsilon_1=\varepsilon_1$ or $A^T\varepsilon_n=\varepsilon_n$. Suppose that $n>2$. We complete the proof that Φ_n is an $(\varepsilon_1, \varepsilon_n)$-set by considering three cases.

CASE 1. $a_{11}=1, a_{nn}\neq 1$. Since 1 is a characteristic value of $(E_{1n}+\varepsilon_1\varepsilon_i^T)A$,

$$\det\begin{bmatrix} a_{11}+a_{i1}-1 & a_{1n}+a_{in} \\ a_{n1} & a_{nn}-1 \end{bmatrix}=0, \qquad i=2,3,\cdots,n-1. \tag{4}$$

Since $a_{n1}=0$, $a_{11}=1$, and $a_{nn}\neq 1$, equation (4) implies that $a_{i1}=0$ for $i=2,3,\cdots,n-1$. Therefore, $A\varepsilon_1=\varepsilon_1$.

CASE 2. $a_{11}\neq 1, a_{nn}=1$. Since 1 is a characteristic value of $(E_{1n}+\varepsilon_j\varepsilon_n^T)A$,

$$\det\begin{bmatrix} a_{11}-1 & a_{1j} & a_{1n} \\ a_{n1} & a_{nj}-1 & a_{nn} \\ a_{n1} & a_{nj} & a_{nn}-1 \end{bmatrix}=0, \qquad j=2,3,\cdots,n-1. \tag{5}$$

Since $a_{n1}=0$, $a_{11}\neq 1$, and $a_{nn}=1$, equation (5) implies that $a_{nj}=0$ for $j=2,3,\cdots,n-1$. Hence, $A^T\varepsilon_n=\varepsilon_n$.

CASE 3. $a_{11}=1=a_{nn}$. Since 1 is a characteristic value of

$$(E_{1n}+\varepsilon_1\varepsilon_i^T+\varepsilon_j\varepsilon_n^T)A,$$

$$\det\begin{bmatrix} a_{11}+a_{i1}-1 & a_{1j}+a_{ij} & a_{1n}+a_{in} \\ a_{n1} & a_{nj}-1 & a_{nn} \\ a_{n1} & a_{nj} & a_{nn}-1 \end{bmatrix}=0 \tag{6}$$

for $i,j=2,3,\cdots,n-1$. Therefore, since $a_{n1}=0$ and $a_{11}=1=a_{nn}$,

$$a_{i1}a_{nj}=0, \qquad i,j=2,3,\cdots,n-1,$$

and we see that $A\varepsilon_1=\varepsilon_1$ or $A^T\varepsilon_n=\varepsilon_n$.

We now show that Ψ_n is minimal. If

$$A = \sum_{k=2}^{n} \varepsilon_k \varepsilon_1^T + \sum_{m=2}^{n-1} \varepsilon_1 \varepsilon_m^T - \sum_{k,m=2}^{n-1} \varepsilon_k \varepsilon_m^T,$$

then it is not difficult to show that $A\varepsilon_1 \neq \varepsilon_1$ and $A^T \varepsilon_n \neq \varepsilon_n$, while equations (3) through (6) hold. This implies that $\Psi_n - \{E_{1n}\}$ is not an $(\varepsilon_1, \varepsilon_n)$-set. If $A = \varepsilon_1 \varepsilon_n^T + \varepsilon_n \varepsilon_1^T$, then $A\varepsilon_1 \neq \varepsilon_1$ and $A^T \varepsilon_n \neq \varepsilon_n$, while equations (2), (4), (5), and (6) hold. This implies that $\Psi_n - \{E_{1n} + \varepsilon_1 \varepsilon_n^T\}$ is not an $(\varepsilon_1, \varepsilon_n)$-set. If $m \in \{2, 3, \cdots, n-1\}$ and $A = \varepsilon_1 \varepsilon_1^T + \varepsilon_m \varepsilon_1^T$, then $A\varepsilon_1 \neq \varepsilon_1$ and $A^T \varepsilon_n \neq \varepsilon_n$, while equations (2) through (6) hold except when $i=m$ in equation (4). This implies that

$$\Psi_n - \{E_{1n} + \varepsilon_1 \varepsilon_m^T\}, \qquad m = 2, 3, \cdots, n-1,$$

is not an $(\varepsilon_1, \varepsilon_n)$-set. If $k \in \{2, 3, \cdots, n-1\}$ and $A = \varepsilon_n \varepsilon_k^T + \varepsilon_n \varepsilon_n^T$, then $A\varepsilon_1 \neq \varepsilon_1$ and $A^T \varepsilon_n \neq \varepsilon_n$, while equations (2) through (6) hold except when $j=k$ in equation (5). This implies that

$$\Psi_n - \{E_{1n} + \varepsilon_k \varepsilon_n^T\}, \qquad k = 2, 3, \cdots, n-1,$$

is not an $(\varepsilon_1, \varepsilon_n)$-set. If $k, m \in \{2, 3, \cdots, n-1\}$ and $A = E_{1n} + \varepsilon_m \varepsilon_1^T + \varepsilon_n \varepsilon_k^T$, then $A\varepsilon_1 \neq \varepsilon_1$ and $A^T \varepsilon_n \neq \varepsilon_n$, while equations (2) through (6) hold except when $i=m$ and $j=k$ in equation (6). This implies that

$$\Psi_n - \{E_{1n} + \varepsilon_1 \varepsilon_m^T + \varepsilon_k \varepsilon_n^T\}, \qquad k, m = 2, 3, \cdots, n-1,$$

is not an $(\varepsilon_1, \varepsilon_n)$-set. Hence, Ψ_n is a minimal $(\varepsilon_1, \varepsilon_n)$-set.

LEMMA 3. *Let $\alpha, \beta, \gamma, \delta \in F^n - \{0\}$, and let P and Q be nonsingular matrices over F such that*

$$(7) \qquad P\gamma = \alpha, \quad P^T \beta = c\delta, \quad Q\alpha = \gamma, \quad Q^T c\delta = \beta,$$

where $c \in F - \{0\}$. If Γ and Γ' are sets of n-square matrices over F such that

$$(8) \qquad \Gamma' = \{PBQ \mid B \in \Gamma\},$$

then Γ' is an (α, β)-set if and only if Γ is a (γ, δ)-set.

PROOF. Suppose that Γ is a (γ, δ)-set, and let A be an n-square matrix over F. Assume that

$$A\alpha = \alpha \quad \text{or} \quad A^T \beta = \beta.$$

Since (7) holds, this implies that $AP\gamma = Q^{-1}\delta$ or $A^T Q^T \delta = (P^T)^{-1}\delta$. Therefore,

$$(QAP)\gamma = \gamma \quad \text{or} \quad (QAP)^T \delta = \delta.$$

Hence, since Γ is a (γ, δ)-set, 1 is a characteristic value of $B(QAP)$ for every $B \in \Gamma$. If $G=PBQ$, then $B(QAP)$ and GA are similar matrices. Therefore, from (8) we see that 1 is a characteristic value of GA for every $G \in \Gamma'$. Reversing this sequence of steps, we see that $A\alpha=\alpha$ or $A^T\beta=\beta$ if 1 is a characteristic value of GA for every $G \in \Gamma'$. Therefore, Γ' is an (α, β)-set. A similar argument shows that Γ is a (γ, δ)-set if Γ' is an (α, β)-set.

It is convenient to express our principal results on minimal (α, β)-sets in two theorems. One of these applies when $\alpha^T\beta\neq 0$, and the other when $\alpha^T\beta=0$.

THEOREM 1. *Let $\alpha, \beta \in F^n$ with $\alpha^T\beta\neq 0$. If $\{\gamma_i, \delta_i | i=1, 2, \cdots, n-1\}$ is a subset of F^n such that $\{\alpha, \gamma_1, \gamma_2, \cdots, \gamma_{n-1}\}$ and $\{\beta, \delta_1, \delta_2, \cdots, \delta_{n-1}\}$ are independent while α and β are orthogonal to δ_i and γ_i, respectively, for $i=1, 2, \cdots, n-1$, then*

$$\Gamma = \{(\alpha^T\beta)^{-1}\alpha\beta^T\} \cup \{(\alpha^T\beta)^{-1}\alpha\beta^T + \gamma_i\delta_j^T \mid i, j = 1, 2, \cdots, n-1\}$$

is a minimal (α, β)-set.

PROOF. Let P be the n-square matrix with columns $\alpha, \gamma_1, \gamma_2, \cdots, \gamma_{n-1}$, and let Q be the n-square matrix with columns $\beta', \delta_1, \delta_2, \cdots, \delta_{n-1}$, where $\beta'=(\alpha^T\beta)^{-1}\beta$. Then P and Q are nonsingular with

$$P\varepsilon_1 = \alpha, \quad P^T\beta = (\alpha^T\beta)\varepsilon_1, \quad Q^T\alpha = \varepsilon_1, \quad Q(\alpha^T\beta)\varepsilon_1 = \beta.$$

Since $\Gamma=\{PBQ^T | B \in \Phi_n\}$, it follows from Lemmas 1 and 3 that Γ is a minimal (α, β)-set.

THEOREM 2. *Let $\alpha, \beta \in F^n-\{0\}$ with $\alpha^T\beta=0$. Suppose that*

$$\{\gamma_i, \delta_i \mid i = 1, 2, \cdots, n-1\}$$

is a subset of F^n such that $\{\alpha, \gamma_1, \gamma_2, \cdots, \gamma_{n-1}\}$ and $\{\beta, \delta_1, \delta_2, \cdots, \delta_{n-1}\}$ are independent, $\alpha^T\delta_1=1=\beta^T\gamma_{n-1}$, and α and β are orthogonal to δ_{i+1} and γ_i, respectively, for $i=1, 2, \cdots, n-2$. If $H=\alpha\delta_1^T+\gamma_{n-1}\beta^T$, then

$$\Gamma = \{H, H + \alpha\beta^T\} \\ \cup \{H + \alpha\delta_{j+1}^T, H + \gamma_i\beta^T, H + \alpha\delta_{j+1}^T + \gamma_i\beta^T \mid i,j = 1,2,\cdots,n-2\}$$

is a minimal (α, β)-set.

PROOF. Let P be the n-square matrix with columns $\alpha, \gamma_1, \gamma_2, \cdots, \gamma_{n-1}$, and let Q be the n-square matrix with columns $\delta_1, \delta_2, \cdots, \delta_{n-1}, \beta$. Then P and Q are nonsingular with

$$P\varepsilon_1 = \alpha, \quad P^T\beta = \varepsilon_n, \quad Q^T\alpha = \varepsilon_1, \quad Q\varepsilon_n = \beta.$$

Since $\Gamma=\{PBQ^T|B\in\Psi_n\}$, it follows from Lemmas 2 and 3 that Γ is a minimal (α,β)-set.

Let $\alpha,\beta\in F^n-\{0\}$. Clearly a subset $\{\gamma_i,\delta_i|i=1,2,\cdots,n-1\}$ of F^n exists satisfying the conditions of Theorems 1 or 2 according to whether $\alpha^T\beta\neq 0$ or $\alpha^T\beta=0$. Hence we have the following.

COROLLARY 1. *If* $\alpha,\beta\in F^n-\{0\}$, *there exists a minimal* (α,β)*-set of cardinality* n^2-2n+2.

If we let $\alpha=\beta=\varepsilon$ in this corollary we obtain the following.

COROLLARY 2. *There exists a set* Γ *of n-square matrices over F of cardinality* n^2-2n+2 *such that, for each n-square matrix A over F, A is stochastic if and only if* 1 *is a characteristic value of PA for every* $P\in\Gamma$.

Consideration of (α,β)-sets was motivated by Brualdi and Wielandt's remark [**1**] on the difficulty of finding sets of fewer than $n!$ permutation matrices which could be used in their characterization of the n-square stochastic matrices. We have no general results on this problem, but we can show that every set of five 3-square permutation matrices is a minimal $(\varepsilon,\varepsilon)$-set.

Let Γ be the set of all 3-square permutation matrices over F, excluding the identity matrix. Let $A=(a_{ij})$ be a 3-square matrix over F. Suppose that 1 is a characteristic value of PA for every $P\in\Gamma$. Then

$$\det(I-PA)=0 \qquad \forall P\in\Gamma.$$

If we multiply each matrix $I-PA$ on the left by P^T, we obtain

$$\det(P-A)=0 \qquad \forall P\in\Gamma. \tag{9}$$

Let

$$\Delta=\left\{\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}.$$

If we add the second and third row of $P-A$ to the first row of $P-A$, and then add the second and third column to the first column we see from (9) that

$$\det\begin{bmatrix}s & c_2 & c_3\\ r_2 & p_{22}-a_{22} & p_{23}-a_{23}\\ r_3 & p_{32}-a_{32} & p_{33}-a_{33}\end{bmatrix}=0 \tag{10}$$

for every matrix $\left[\begin{smallmatrix}p_{22}&p_{23}\\p_{32}&p_{33}\end{smallmatrix}\right]$ in Δ, where

$$s=3-\sum_{i,j=1}^{3}a_{ij}, \qquad c_k=1-\sum_{i=1}^{3}a_{ik}, \qquad r_k=1-\sum_{j=1}^{3}a_{kj}, \qquad k=2,3.$$

Suppose that $s \neq 0$. Then (10) must hold for $c_2 = c_3 = 0$, since any multiple of the first column can be added to any other column without changing the determinant. If $s \neq 0$ and $c_2 = c_3 = 0$ in (10), we see that

$$
\begin{aligned}
a_{22}a_{33} - (a_{23} - 1)(a_{32} - 1) &= 0, \\
a_{22}a_{33} - (a_{23} - 1)a_{32} &= 0, \\
a_{22}a_{33} - a_{23}(a_{32} - 1) &= 0, \\
(a_{22} - 1)a_{33} - a_{23}a_{32} &= 0, \\
a_{22}(a_{33} - 1) - a_{23}a_{32} &= 0.
\end{aligned} \tag{11}
$$

It is not difficult to show that this system of equations has no solution. Hence, $s = 0$.

Since $s = 0$, from (10) we obtain a homogeneous system of five linear equations in the four unknowns c_2r_2, c_2r_3, c_3r_2, c_3r_3. Since the coefficient matrix for this system has rank four, we see that

$$r_2 = r_3 = 0 \quad \text{or} \quad c_2 = c_3 = 0.$$

Therefore, since $s = 0$, $A\varepsilon = \varepsilon$ or $A^T\varepsilon = \varepsilon$. Therefore, Γ is an $(\varepsilon, \varepsilon)$-set. If any equation is removed from system (11), then the remaining system has a solution. Hence Γ is a minimal $(\varepsilon, \varepsilon)$-set. It now follows from Lemma 3 that every set of five 3-square permutation matrices is a minimal $(\varepsilon, \varepsilon)$-set.

We have determined by machine calculation that the 12 matrices

$$
\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix},
\begin{bmatrix} 1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0 \end{bmatrix},
\begin{bmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0 \end{bmatrix},
\begin{bmatrix} 0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0 \end{bmatrix},
$$

$$
\begin{bmatrix} 0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0 \end{bmatrix},
\begin{bmatrix} 0&0&0&1\\0&0&1&0\\1&0&0&0\\0&1&0&0 \end{bmatrix},
\begin{bmatrix} 0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1 \end{bmatrix},
\begin{bmatrix} 0&0&1&0\\0&0&0&1\\0&1&0&0\\1&0&0&0 \end{bmatrix},
$$

$$
\begin{bmatrix} 0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0 \end{bmatrix},
\begin{bmatrix} 0&0&1&0\\0&1&0&0\\0&0&0&1\\1&0&0&0 \end{bmatrix},
\begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix},
\begin{bmatrix} 0&0&1&0\\1&0&0&0\\0&0&0&1\\0&1&0&0 \end{bmatrix}
$$

over $GF(2)$ form a minimal $(\varepsilon, \varepsilon)$-set. There also exist minimal $(\varepsilon, \varepsilon)$-sets of 4-square permutation matrices over $GF(2)$ of cardinality 13, but there exist none of cardinality less than 12.

Reference

1. R. A. Brualdi and H. W. Wielandt, *A spectral characterization of stochastic matrices*, Linear Algebra and Appl. **1** (1968), no. 1, 65–71. MR **36** #6435.

Department of Mathematics, University of Alabama, Huntsville, Alabama 35807 (Current address of P. M. Gibson)

Current address (E. T. Beasley): Department of Computer Science, Ohio State University, Columbus, Ohio 43210