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A CHARACTERIZATION OF HILBERT SPACE

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ABSTRACT. A real Banach space E of dimension $\geqq 3$ is an inner product space iff there exists a bounded smooth convex subset of E which is the range of a nonexpansive retraction.

De Figueiredo and Karlovitz [3] have shown that if E is a strictly convex real finite-dimensional Banach space and $\dim E \geqq 3$ then there can exist no bounded smooth nonexpansive retract of E unless E is a Hilbert space. (A subset F of E is a nonexpansive retract of E if it is the range of a nonexpansive retraction $r: E \to F$.) This is a consequence of their more general result that if E is reflexive and a convex nonexpansive retract of E has at a boundary point x_0 a unique supporting hyperplane $x_0 + H$ then H is the range of a projection of norm 1. As they have pointed out, the latter theorem fails in nonreflexive spaces (the unit ball of $C[0, 1]$ furnishes a counterexample). Nevertheless, their first result is true in general:

THEOREM. *Suppose E is a real Banach space with* $\dim E \geqq 3$. *Then E is an inner product space iff there exists a bounded smooth nonexpansive retract of E with nonempty interior.*

We separate out of the proof of the theorem a lemma, valid in all real Banach spaces:

LEMMA. *Suppose F is a bounded smooth closed convex subset of a real Banach space E and F has nonempty interior. Then given disjoint bounded closed convex sets M and K in E with K compact, there exist $p \in E$ and $\lambda > 0$ such that $K \subset p + \lambda F$ and $(p + \lambda F) \cap M = \varnothing$.*

PROOF OF LEMMA. Clearly the hypotheses and conclusions of the lemma are invariant if K and M are translated by the same vector; thus without loss of generality we may assume $0 \in K$. Similarly, we may also assume $0 \in \operatorname{int} F$. Since K is compact and M is closed, a basic separation theorem for convex sets assures the existence of a closed hyperplane H which strictly separates M and K; that is, there exist $w \in E^*$, $c \in R^1$

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such that $H=\{x \in E: w(x)=c\}$ and

$$(1) \qquad 0 \leqq \sup\{w(y): y \in K\} < c < \inf\{w(y): y \in M\}.$$

Since K and M are bounded, (1) also holds for functionals sufficiently close to w (in the norm of E^*). By the Bishop-Phelps theorem **[1]** the support functionals of F are dense in E^*; thus we may assume without loss of generality that the functional w in (1) is a support functional of F.

If w supports F at x_0, then $H=\{x: w(x)=c\}$ is the tangent hyperplane to μF at μx_0, where $\mu=c/w(x_0)>0$. Let $F_t=(1-t)\mu x_0+t\cdot \operatorname{int}(\mu F)$ for $t>0$. Since $\operatorname{int} \mu F$ is convex and μx_0 is a boundary point of μF, it is easily seen that $F_s \subset F_t$ if $s<t$.

The family $\{F_t: t>0\}$ is an open cover of K. (In fact, since μF is smooth at μx_0, it is easily verified that $\bigcup_{t>0} F_t$ is the open half-space $\{x: w(x)<c\}$ with boundary H which includes $\operatorname{int}(\mu F)$. By (1), K is a subset of this open half-space.)

Since the cover $\{F_t: t>0\}$ is linearly ordered by inclusion and K is compact, there exists $t>0$ such that $K \subset F_t \subset \mathrm{Cl}(F_t)$. On the other hand, $M \cap \mathrm{Cl}(F_t)=\varnothing$ because M is a subset of the opposite open half-space $\{x: w(x)>c\}$. Since $\mathrm{Cl}(F_t)=(1-t)\mu x_0+\mu t F$, we may take $p=(1-t)\mu x_0$, $\lambda=\mu t$ to reach the conclusion of the lemma. Q.E.D.

Proof of Theorem. Necessity is trivial, since it is well known that the closed unit ball of a Hilbert space E is a smooth nonexpansive retract of E. (In fact, every closed convex subset of E is a nonexpansive retract of E—the proximity mapping is a nonexpansive retraction.)

To prove sufficiency, let E_1 be any three-dimensional subspace of E, E_0 any two-dimensional subspace of E_1, and x_0 any point of $E_1 \backslash E_0$. Fix $R>0$ and define

$$K = \{x \in E_0: \|x\| \leqq R\},$$
$$M = \{x \in E: \|x - y\| \leqq \|x_0 - y\| \text{ for all } y \in K\}.$$

Then K and M are bounded closed convex sets with K compact. We claim that $K \cap M \neq \varnothing$.

Otherwise, by the lemma there exist $p \in E$ and $\lambda>0$ such that $K \subset p+\lambda F$ and $(p+\lambda F) \cap M=\varnothing$. If f is a nonexpansive retraction of E onto F, it is easily verified that $g: x \mapsto \lambda f(\lambda^{-1}(x-p))+p$ is a nonexpansive retraction of E onto $p+\lambda F$. In particular, for any $y \in K \subset p+\lambda F$ we have $g(y)=y$ so

$$\|g(x_0) - y\| = \|g(x_0) - g(y)\| \leqq \|x_0 - y\|;$$

by definition, $g(x_0)$ therefore belongs to M. But $g(x_0) \in p+\lambda F$ since g retracts E onto $p+\lambda F$. This is a contradiction since $(p+\lambda F) \cap M=\varnothing$.

We have shown that $K \cap M \neq \varnothing$ so that for each $R>0$ there exists $x_R \in E_0$ with

$$\|x_R - y\| \leqq \|x_0 - y\| \tag{2}$$

for all y in E_0 with $\|y\| \leqq R$. Since (2) holds for $y=0$ in particular, the set $\{x_R : R>0\}$ is bounded and (since $\dim E_0=2$) therefore relatively compact. Hence there exists a sequence $R_n \to \infty$ such that $x_{R_n} \to x_\infty$ for some x_∞ in E_0. It follows from (2) that

$$\|x_\infty - y\| \leqq \|x_0 - y\| \quad \text{for all } y \in E_0. \tag{3}$$

As in Kakutani [4], (3) implies the existence of a linear projection P of E_1 onto E_0 with $\|P\|=1$.

To summarize: Whenever E_1 is a three-dimensional subspace of E and E_0 is a two-dimensional subspace of E_1, then there exists a projection P of E_1 onto E_0 with $\|P\|=1$. By Kakutani [**4**], E_1 must be an inner product space. Finally, since every three-dimensional subspace of E is an inner product space, E itself must be an inner product space. Q.E.D.

REMARK. The same technique can be used to prove the following variant of Kakutani's theorem: If E is a real Banach space of dimension $\geqq 3$ and every closed linear subspace of E of codimension 1 is the range of a projection of norm 1, then E is an inner product space. It is only necessary to show that every closed half-space in E is a nonexpansive retract of E; and this can be done as in Bruck [**2**, Theorem 5].

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