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TENSOR PRODUCTS AND ALMOST PERIODICITY[1]

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ABSTRACT. Let E and F be locally convex spaces and G their completed ε-tensor product. It is shown that if S and T are weakly almost periodic equicontinuous semigroups of operators on E and F respectively, then, under mild restrictions on E or F, $S\otimes T$ is a weakly almost periodic equicontinuous semigroup of operators on G, and the almost periodic and flight vector subspaces of G are related in a natural way to the corresponding subspaces of E and F via the ε-tensor product. Furthermore, if E and F both decompose into a direct sum of these subspaces then so does G.

1. **Weakly almost periodic semigroups.** Let E be a locally convex (Hausdorff) linear topological space with topological dual E', and let $L(E)$ denote the space of continuous linear operators on E. A *semigroup of operators* on E is a subset S of $L(E)$ containing the identity operator and closed under composition. A vector $x\in E$ is said to be *weakly* (*strongly*) *almost periodic* under S if its orbit $Sx=\{ux:u\in S\}$ is relatively compact in the weak (strong) topology of E. The set of all weakly (strongly) almost periodic vectors in E shall be denoted by $W(E,S)$ ($A(E,S)$). Occasionally we shall suppress the symbols E or S from the notation if they are understood from context. If $W(E,S)=E$ ($A(E,S)=E$) we say that S is *weakly* (*strongly*) *almost periodic*.

It is easily seen that multiplication in a semigroup of operators S on a locally convex space E is separately continuous with respect to the weak or strong operator topologies on $L(E)$, that is to say S is a topological semigroup. Moreover if S is equicontinuous then multiplication is actually jointly continuous in the strong operator topology. The following lemma is at the heart of the theory of weak almost periodicity. A proof can be found in [**1**].

LEMMA 1.1. *Let S be a weakly almost periodic equicontinuous semigroup of operators on a locally convex space E, and let $\bar{S}$ denote the closure*

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of S in the weak operator topology of $L(E)$. Then $\bar{S}$ is a compact topological semigroup in the weak operator topology.

In connection with Lemma 1.1 we remark that if E is barreled or a Baire space then the weak almost periodicity of S implies equicontinuity [**8**, p. 83], and if E is semireflexive then the converse implication holds [**8**, p. 144].

Using Lemma 1.1 the following generalization of a result of Eberlein [**3**] is easily proved (see [**1**]).

THEOREM 1.2. *Let S be an equicontinuous semigroup of operators on a locally convex space E. Then $W=W(E, S)$ is an S-invariant linear subspace of E. Moreover, if E is complete then W is closed.*

COROLLARY 1.3. *If S is a weakly almost periodic equicontinuous semigroup on E, then the extension of S to the completion of E is a weakly almost periodic equicontinuous semigroup.*

2. **Tensor product of weakly almost periodic semigroups.** Let E and F be locally convex spaces and $G=E \otimes_\varepsilon F$ the completion in the ε-topology of the tensor product $E\otimes F$. If S and T are semigroups of operators on E and F respectively we shall let $S\otimes T$ denote the set of all operators in $L(G)$ of the form $u\otimes v$, where $u \in S$ and $v \in T$. Recall that $u\otimes v$ is defined by the equation $(u\otimes v)(x\otimes y)=ux\otimes vy$. It follows easily that $S\otimes T$ is a semigroup of operators on G, where $(u_1\otimes v_1)(u_2\otimes v_2)=u_1u_2\otimes v_1v_2$. Furthermore, if S and T are equicontinuous then so is $S\otimes T$ [**5**].

Our main object in this section is to determine conditions on E and F under which the weak almost periodicity of S and T implies that of $S\otimes T$. To this end we require the following lemma.

LEMMA 2.1. *Let E and F be locally convex spaces, A and B relatively weakly compact subsets of E and F respectively. Suppose one of the following conditions holds*:

(i) *E or F has separable dual*;

(ii) *E or F is a Banach space*;

(iii) *A or B is relatively strongly compact.*

Then $A\otimes B=\{x\otimes y : x \in A, y \in B\}$ is relatively compact in the weak topology of $G=E \otimes_\varepsilon F$.

PROOF. By the completeness of G it suffices to show that any sequence $(x_n\otimes y_n)$ in $A\otimes B$ has a weak cluster point [**8**, p. 187]. Let (x_α) be a subnet of (x_n) converging weakly to $x_0 \in E$ and (y_α) a subnet of (y_n) converging weakly to $y_0 \in F$. We shall show that $x_\alpha\otimes y_\alpha$ converges weakly to $x_0\otimes y_0$ in G. The equality $x_\alpha\otimes y_\alpha-x_0\otimes y_0=(x_\alpha-x_0)\otimes y_0+x_\alpha\otimes(y_\alpha-y_0)$ and the separate weak continuity of tensor product show that it suffices to prove that $x_\alpha \otimes (y_\alpha - y_0)$ converges weakly to zero. Let $\varphi \in G'$ and suppose $\varphi(x_\alpha \otimes (y_\alpha - y_0))$ does not converge to zero. Then there exist a positive number ε and a subnet $x_\beta \otimes (y_\beta - y_0)$ such that $|\varphi(x_\beta \otimes (y_\beta - y_0))| \geqq \varepsilon$ for all β. Now for all $x \in E$, $y \in F$,

$$\varphi(x \otimes y) = \int_{A' \times B'} \langle x, x' \rangle \langle y, y' \rangle \, d\mu(x', y')$$

where A' and B' are equicontinuous subsets of E' and F' respectively and μ is a Borel measure on $A' \times B'$ with total variation $|\mu| \leqq 1$ [**8**, p. 168]. Since A is bounded, $\lambda = \sup\{|\langle x, x' \rangle| : x \in A, x' \in A'\} < \infty$. Therefore for all β we have

$$\text{(1)} \qquad \int_{A' \times B'} |\langle y_\beta - y_0, y' \rangle| \, d|\mu| (x', y') \geqq \varepsilon / \lambda > 0.$$

Suppose F' is separable with total set $\{y'_n\}$, and choose a sequence (y_k) from the set $\{y_\beta\}$ such that $\lim_{k \to \infty} \langle y_k - y_0, y'_n \rangle = 0$ for every n. Since B is bounded, it follows easily that y_k converges weakly to y_0. By Lebesgue's Dominated Convergence Theorem we thus obtain a contradiction to (1).

Now assume F is a Banach space. Then by Eberlein's Theorem we may suppose y_n converges weakly to y_0 and in the same manner as above we contradict (1).

Finally, if B is relatively strongly compact then we may assume $\langle y_\beta - y_0, y' \rangle$ converges to 0 uniformly on B' and again we contradict (1). Q.E.D.

We may now state and prove the main result of this section.

THEOREM 2.2. *Let E and F be locally convex spaces, S and T weakly almost periodic equicontinuous semigroups of operators on E and F respectively, and let $G = E \otimes_\varepsilon F$. Suppose further that one of the following conditions holds*:

(i) *E or F has separable dual*;

(ii) *E or F is a Banach space*;

(iii) *S or T is strongly almost periodic.*

Then $S \otimes T$ is a weakly almost periodic equicontinuous semigroup of operators on G, and $\mathrm{Cl}(S \otimes T) = \bar{S} \otimes \bar{T}$ *(closures taken in the weak operator topologies).*

PROOF. If $x \in E$ and $y \in F$, then $S \otimes T(x \otimes y) = Sx \otimes Ty$; hence if any of the conditions (i)–(iii) holds, Lemma 2.1 implies that $x \otimes y \in W(G, S \otimes T)$. By Theorem 1.2, $G \subset W(G)$, i.e., $S \otimes T$ is weakly almost periodic.

Let $w \in \mathrm{Cl}(S \otimes T)$, (u_α) and (v_α) nets in S and T respectively such that $u_\alpha \otimes v_\alpha$ converges to w in the weak operator topology of $L(G)$. We may assume u_α converges to $u \in \bar{S}$ and v_α converges to $v \in \bar{T}$. Let $x \in E$, $y \in F$, $x' \in E'$, $y' \in F'$. Then $x' \otimes y'(u_\alpha \otimes v_\alpha(x \otimes y)) = \langle u_\alpha x, x' \rangle \langle v_\alpha y, y' \rangle$ converges to $\langle ux, x' \rangle \langle vy, y' \rangle$, hence $x' \otimes y'(u \otimes v(x \otimes y) - w(x \otimes y)) = 0$. If $\varphi \in G'$, then by definition of the ε-topology $|\varphi(\theta)| \leqq \sup\{|x' \otimes y'(\theta)| : x' \in A', y' \in B'\}$ for all $\theta \in G$, where A' and B' are equicontinuous subsets of E' and F' respectively [5]. It follows that $\varphi(u \otimes v(x \otimes y) - w(x \otimes y)) = 0$ and hence that $w = u \otimes v \in \bar{S} \otimes \bar{T}$.

Conversely, let $u \otimes v \in \bar{S} \otimes \bar{T}$ and let (u_α) and (v_α) be as before. For fixed β, $u \otimes v_\beta$ is the weak operator limit of $u_\alpha \otimes v_\beta$ and is therefore a member of $\mathrm{Cl}(S \otimes T)$. Taking the limit with respect to β we see that $u \otimes v \in \mathrm{Cl}(S \otimes T)$. Q.E.D.

3. **Decomposition of** $E \otimes_\varepsilon F$. In this section we shall determine conditions under which $E \otimes_\varepsilon F$ has a direct sum decomposition into subspaces of almost periodic and flight vectors.

If S is a weakly almost periodic semigroup of operators on the locally convex space E, we denote by E_r (E_0) the set of all vectors $x \in E$ having the property that $\mathrm{Cl}(Sx) = \mathrm{Cl}(Sy)$ for all $y \in \mathrm{Cl}(Sx)$ ($0 \in \mathrm{Cl}(Sx)$), where the closures are in the weak topology of E. E_r is the set of *reversible vectors* of E, E_0 the set of *flight vectors* [6]. Also, we shall let E_p denote the closed linear span of all finite-dimensional S-invariant subspaces H of E which have the property that S restricted to H is contained in an equicontinuous (i.e., uniformly bounded) group of operators on H. E_p is the set of *almost periodic vectors* [2].

The proofs of the following theorems rely heavily on the ideal theory of compact topological semigroups as developed by deLeeuw and Glicksberg in [2]. In particular we shall make use of the fact that a compact topological semigroup R contains a smallest (nonempty) two-sided ideal $K(R)$, called the *kernel* of R, and that $K(R)$ contains at least one idempotent element.

We shall also need the analogs of Theorems 4.9, 4.10 and 4.11 of [2] in the setting of locally convex spaces. An examination of the proofs of these theorems reveals the following: Theorem 4.9 holds for any locally convex space and Theorems 4.10, 4.11 hold for quasi-complete spaces. For the details the interested reader is referred to [7]. These theorems may also be formulated so that no reference to topology need be made [1].

THEOREM 3.1. *Let S and T be weakly almost periodic equicontinuous semigroups of operators on the locally convex spaces E and F respectively. If $S \otimes T$ is weakly almost periodic on $G = E \otimes_\varepsilon F$ and if E_0 and F_0 are closed invariant linear spaces, then G_0 is a closed invariant linear subspace of G and is the closure of $E_0 \otimes F + E \otimes F_0$.*

PROOF. By Theorem 4.9 of [**2**] $\bar{S}$ and $\bar{T}$ have unique minimal left ideals I and J respectively. To show G_0 is a closed invariant subspace of G it suffices by the same theorem to show that $\mathrm{Cl}(S\otimes T)$ $(=\bar{S}\otimes\bar{T})$ has a unique minimal left ideal, namely $I\otimes J$.

It is clear that $I\otimes J$ is a left ideal of $\mathrm{Cl}(S\otimes T)$. To show that it is minimal let K be a left ideal of $\mathrm{Cl}(S\otimes T)$ contained in $I\otimes J$, and choose any $u_0\otimes v_0 \in K$ such that $u_0 \in I$, $v_0 \in J$. Now, by Corollary 2.4 of [**2**], $I=K(\bar{S})$, and by Theorem 2.3 [**2**], $Iu_0=I$. Hence if e is any projection in $K(\bar{S})$, then there exists $u \in I$ such that $uu_0=e$. Similarly, if f is a projection in $K(\bar{T})$, there exists $v \in J$ such that $vv_0=f$. Since K is a left ideal, $e\otimes f=(u\otimes v)(u_0\otimes v_0) \in K$. Fix e and f and let $I_1=\{u \in I: u\otimes f \in K\}$. I_1 is easily seen to be a non-empty left ideal of $\bar{S}$, hence $I_1=I$, i.e., $u\otimes f \in K$ for every $u \in I$. Now let $J_1=\{v \in J: eu\otimes v \in K\}$, where u is a fixed element of I. J_1 is a left ideal of J, and, by what has just been proved, J_1 contains f. Therefore $J_1=J$, and we have shown that $eu\otimes v \in K$ for all $u \in I$, $v \in J$ and all projections $e \in I$. By Corollary 2.4 of [**2**], I is the union of all right ideals $e\bar{S}$, where $e^2=e \in I$. Hence given any $u \in I$ there exists a projection $e \in I$ such that $eu=u$, and it follows from above that $I\otimes J=K$.

By Theorem 2.3 of [**2**], $K(\mathrm{Cl}(S\otimes T))$ is the union of all minimal left ideals of $\mathrm{Cl}(S\otimes T)$ and therefore contains $I\otimes J=K(\bar{S})\otimes K(\bar{T})$. But the latter is a two-sided ideal and so must equal $K(\mathrm{Cl}(S\otimes T))$. Thus $I\otimes J$ contains all minimal left ideals and therefore must be the unique minimal left ideal of $\mathrm{Cl}(S\otimes T)$.

Now let $\theta=\lim_\alpha \theta_\alpha \in G_0$, where (θ_α) is a net in $E\otimes F$. By Lemma 4.2 of [**2**] there exists a projection $g \in K(\mathrm{Cl}(S\otimes T))=K(\bar{S})\otimes K(\bar{T})$ such that $g(\theta)=0$. Let $e \in K(\bar{S})$ and $f\in K(\bar{T})$ be arbitrary projections. Then $e\otimes f$ is a projection in $K(\mathrm{Cl}(S\otimes T))$, hence $(e\otimes f)g=e\otimes f$ by Corollary 2.4 [**2**]. In particular, $e\otimes f(\theta)=0$, hence $\theta=\lim_\alpha(\theta_\alpha-e\otimes f(\theta_\alpha))$. For a fixed $\theta_\alpha=\sum_{i=1}^n x_i\otimes y_i$,

$$\begin{aligned}\theta_\alpha - e\otimes f(\theta_\alpha) &= \sum (x_i - ex_i)\otimes y_i \\ &\quad + \sum ex_i \otimes (y_i - fy_i) \in E_0\otimes F + E\otimes F_0,\end{aligned}$$

so $\theta \in \mathrm{Cl}(E_0\otimes F+E\otimes F_0)$. Therefore we have $G_0\subset \mathrm{Cl}(E_0\otimes F+E\otimes F_0)$. The reverse inclusion follows readily from the fact that G_0 is a closed subspace of G. Q.E.D.

THEOREM 3.2. *Let S and T be weakly almost periodic equicontinuous semigroups of operators on the quasi-complete locally convex spaces E and F respectively, and suppose $S\otimes T$ is weakly almost periodic on $G=E\otimes_\varepsilon F$. If $E_r=E_p$ and $F_r=F_p$, then $G_r=G_p=E_p\otimes_\varepsilon F_p$.*

PROOF. The hypotheses imply that $\bar{S}$ and $\bar{T}$ have unique minimal right ideals I and J respectively [**2**, Theorem 4.10]. By methods analogous to those used in the proof of Theorem 3.1, $I\otimes J$ is the unique minimal right ideal of $\mathrm{Cl}(S\otimes T)$. Hence by Theorem 4.10 [**2**], $G_r=G_p$.

If $x\in E_p$ and $y\in F_p$, there exist projections $e\in K(\bar{S})$, $f\in K(\bar{T})$ such that $ex=x$ and $fy=y$ [**2**, Lemma 4.1]. Then $e\otimes f$ is a projection in $K(\mathrm{Cl}(S\otimes T))=K(\bar{S})\otimes K(\bar{T})$, and the same lemma shows that $x\otimes y\in G_p$. Thus $E_p\otimes_\varepsilon F_p\subset G_p$. Conversely, let $\theta=\lim_\alpha \theta_\alpha\in G_p$, $\theta_\alpha\in E\otimes F$. Choose a projection $g\in K(\mathrm{Cl}(S\otimes T))$ such that $g\theta=\theta$. If e and f are arbitrary projections in $K(\bar{S})$ and $K(\bar{T})$ respectively, then $e\otimes f$ is a projection in $K(\mathrm{Cl}(S\otimes T))$ and by Corollary 2.4 [**2**], $(e\otimes f)g=g$. It follows that $\theta=e\otimes f(\theta)=\lim_\alpha e\otimes f(\theta_\alpha)$. If $\theta_\alpha=\sum_{i=1}^n x_i\otimes y_i$, then $e\otimes f(\theta_\alpha)=\sum ex_i\otimes fy_i\in E_p\otimes F_p$, hence $\theta\in E_p\otimes_\varepsilon F_p$. Therefore $G_p=E_p\otimes_\varepsilon F_p$. Q.E.D.

We may now prove the main result of this section.

THEOREM 3.3. *If all the hypotheses of Theorems* 3.1 *and* 3.2 *are satisfied, then* $G=G_p\oplus G_0$, *where* $G_p=E_p\otimes_\varepsilon F_p$ *and* $G_0=\mathrm{Cl}(E_0\otimes F+E\otimes F_0)$.

PROOF. By Theorem 4.11 [**2**], $K(\bar{S})$ and $K(\bar{T})$ are compact topological groups; to show $G=G_p\oplus G_0$ it suffices by the same theorem to show that $K(\mathrm{Cl}(S\otimes T))$ is a compact topological group. By Ellis' Theorem [**4**] we need only show that $K(\mathrm{Cl}(S\otimes T))$ is algebraically a group. But this is immediate from the equality $K(\mathrm{Cl}(S\otimes T))=K(\bar{S})\otimes K(\bar{T})$ (see proof of Theorem 3.1). Q.E.D.

The above results may be used in a variety of ways to generate nontrivial examples of weakly almost periodic semigroups of operators with the decomposition property of Theorem 3.3. As an illustration, let E and F be reflexive Banach spaces and let S and T be bounded. Then S and T are obviously weakly almost periodic, hence, according to Theorem 2.2, so is $S\otimes T$. Since $G=E\otimes_\varepsilon F$ need not be reflexive [**9**], this result is decidedly nontrivial. Furthermore, if, say, E and E' are strictly convex and T is commutative, then E and F both have direct sum decompositions into almost periodic and flight vector subspaces [**2**], and therefore, by Theorem 3.3, so does G.

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