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PRODUCTS OF RC-PROXIMITY SPACES

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ABSTRACT. It is shown that the proximity product of RC-proximity spaces need not be an RC-proximity space. This answers a question of Harris. A positive answer would have implied that the product of regular-closed spaces is regular-closed.

1. **Introduction.** D. Harris [**4**] has characterized those spaces which can be densely embedded in a regular-closed space as those regular spaces which admit a compatible generalized proximity, which he called an RC-proximity. In [**4**] he asked five questions. The first and third have since been answered by Sharma and Naimpally [**9**] while Hunsaker and Sharma have recently solved the second by giving necessary and sufficient conditions under which a map from an RC-regular space to a regular-closed space can be extended to the Harris extension. (The following variant of Harris' second problem is still open: characterize those topological spaces X for which any continuous function from X to any regular-closed space can be extended to the Harris extension.) The purpose of this note is to give a negative answer to part of the fourth problem which asked if the product of RC-proximities with the usual definition of product proximity (see Čech [**2**] or Willard [**11**]) is an RC-proximity. A positive answer to this question would have implied that the product of regular-closed spaces is regular-closed [**1**, problem 5]. We shall prove that if (X, δ) is an absolutely closed RC-proximity space which is not compact, then there is a compact Hausdorff space Y for which $X\times Y$ with the product structure is not an RC-proximity space. Our proof will make use of a recent result of Willard [**10**].

Proximity, unmodified, will refer to the usual proximity. A proximally continuous function is called a p-map. Otherwise, all notation and terminology will follow Harris [**4**].

2. **Preliminary definitions and lemmas.**

2.1. DEFINITION. (See Willard [**11**].) If $\{(X_\alpha, \delta_\alpha)\}_\alpha$ is a collection of sets with binary relations then the product structure δ on $X=\prod X_\alpha$ is

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defined as follows:

$A\,\delta\,B$ iff whenever $A=A_1\cup\cdots\cup A_n$ and $B=B_1\cup\cdots\cup B_m$ then there is some A_i and B_j for which $\pi_\alpha(A_i)\,\delta_\alpha\,\pi_\alpha(B_j)$ for all α.

When each $(X_\alpha, \delta_\alpha)$ is a proximity space then δ is the product proximity and is the categorical product.

2.2. LEMMA. *If $\{(X_\alpha, \delta_\alpha)\}_\alpha$ is a collection of R-proximity spaces, then $X=\prod X_\alpha$ with the product structure δ is an R-proximity space. δ is the coarsest R-proximity on X for which each π_α is a p-map.*

PROOF. It follows from Leader's result in [6] that P1–P4 of Harris' definition are satisfied and it is sufficient to verify P5. Let $x\,\not\delta\,A$. We shall find a set $B\subseteq X$ for which $x\,\not\delta\,(X-B)$ and $B\,\not\delta\,A$. This will clearly satisfy P5. Since $x\,\not\delta\,A$, there is a finite decomposition of A, $A=A_1\cup\cdots\cup A_n$ such that for each i, there is some α_i for which $\pi_{\alpha_i}(x)\,\not\delta_{\alpha_i}\,\pi_{\alpha_i}(A_i)$. Since each X_{α_i} is an R-proximity space, for each i there is some $K_i\subseteq X_{\alpha_i}$ such that $\pi_{\alpha_i}(x)\,\not\delta_{\alpha_i}\,(X_{\alpha_i}-K_i)$ and $K_i\,\not\delta_{\alpha_i}\,\pi_{\alpha_i}(A_i)$. Let $B=\bigcap_i \pi_{\alpha_i}^{-1}(K_i)$. Then $X-B=\bigcup_i \pi_{\alpha_i}^{-1}(X_{\alpha_i}-K_i)$. Now, $x\,\not\delta\,\pi_{\alpha_i}^{-1}(X_{\alpha_i}-K_i)$ for each i implies that $x\,\not\delta\,(X-B)$, and since $A_i\,\not\delta\,\pi_{\alpha_i}^{-1}(K_i)$, $A_i\,\not\delta\,B$ for each i, so that $A\,\not\delta\,B$. That δ is the coarsest such structure follows easily from the fact that if Z is an R-proximity space then $f{:}Z\to X$ is a p-map iff $\pi_\alpha\circ f$ is a p-map for all α.

2.3. LEMMA (WILLARD [10]). *If X is a topological space such that for every compact Hausdorff space Y, $\pi_Y{:}X\times Y\to Y$ is a closed map, then X is compact.*

The next lemma is a special case of Theorem 7 of [6].

2.4. LEMMA. *Let (X_1, δ_1) and (X_2, δ_2) be R-proximity spaces and let δ be the product R-proximity. Then $y\,\delta_i\,\pi_i(A)$ implies $\pi_i^{-1}(y)\,\delta\,A$ for $i=1$ or 2.*

PROOF. We shall prove this for $i=2$. Let $y\,\delta_2\,\pi_2(A)$ and assume $\pi_2^{-1}(y)\,\not\delta\,A$. Then there are finite decompositions of A and $\pi_2^{-1}(y)$, $A=A_1\cup\cdots\cup A_n$ and $\pi_2^{-1}(y)=C_1\cup\cdots\cup C_m$ and for each A_i and C_j either $\pi_1(A_i)\,\delta_1\,\pi_1(C_j)$ or $\pi_2(A_i)\,\delta_2\,\pi_2(C_j)$. Now, some member of the decomposition of A, say A_k, must satisfy $\pi_2(A_k)\,\delta_2\,y$. Since $\pi_2^{-1}(y)\,\not\delta\,A$, it follows from the definition of the product that $\pi_1(A_k)\,\not\delta_1\,\pi_1(C_j)$ for all j. But then $\pi_1(A_k)$ is separated from $\bigcup_j \pi_1(C_j)$ and $\bigcup_j \pi_1(C_j)=\pi_1(\bigcup_j C_j)=\pi_1\pi_2^{-1}(y)=X_1$. This is clearly impossible.

3. **Main theorem.**

3.1. THEOREM. *If (X_1, δ_1) is an absolutely closed RC-proximity space which is not compact then there exists a compact Hausdorff space X_2 with* *its unique compatible proximity δ_2 for which $X_1 \times X_2$ with the product R-proximity is not an RC-proximity.*

PROOF. Let (X_1, δ_1) be as in the statement of the theorem. If the result does not hold then for each compact Hausdorff space X_2, the product R-proximity δ on $X_1 \times X_2$ is an RC-proximity. By the theorem of Scarborough and Stone [**8**], $X_1 \times X_2$ with the product topology is regular-closed and hence by [**4**] has exactly one RC-proximity which induces its topology. This RC-proximity is given by $A\ \delta_0\ B$ iff $\bar{A} \cap \bar{B} \neq \varnothing$. It follows from our assumption that $\delta = \delta_0$. We claim that $\pi_2 : X_1 \times X_2 \rightarrow X_2$ is closed. Let F be a closed subset of $X_1 \times X_2$ and let $y \in \mathrm{Cl}((\pi_2(F))$. That is, let $y\ \delta_2\ \pi_2(F)$. By Lemma 2.4, $\pi_2^{-1}(y)\ \delta\ F$, so that $\pi_2^{-1}(y)\ \delta_0\ F$. Since both $\pi_2^{-1}(y)$ and F are closed, $\pi_2^{-1}(y) \cap F \neq \varnothing$, from which it follows that $y \in \pi_2(F)$ and $\pi_2(F)$ is closed. Since this is true for any compact Hausdorff space X_2, Lemma 2.3 implies that X_1 is compact. This contradiction establishes the theorem.

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