PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 43, Number 1, March 1974

ON SUM-FREE SUBSEQUENCES[1]

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ABSTRACT. A sequence of real numbers is said to be *sum-free* if no number of the sequence is the sum of distinct elements of the same sequence. In this paper we show that a sequence S of n positive real numbers has a sum-free subsequence containing at least $(2n)^{1/2}-\log_2(4n)$ elements.

Choi [1], studying a problem of Erdös [2], has proven that a sequence of n positive real numbers has a sum-free subsequence of $\geqq(36/35)n^{1/2}$ elements. The purpose of this paper is to show that the constant 36/35 can be improved to $2^{1/2}-\varepsilon$. In fact, more precisely, we have

THEOREM. *A sequence S of n positive real numbers has a sum-free subsequence containing $\geqq(2n)^{1/2}-\log_2(4n)$ elements.*

PROOF. Let a_1 be the least element of S; let a_2 be the least element of S which is $\geqq 2a_1$, and inductively choose a_{i+1} to be the least element of S which is $\geqq 2a_i$. This yields a finite sequence $a_1<a_2<\cdots<a_m$ and every element of S is $<2a_m$. For $1\leqq i\leqq m-1$, let T_i be the subsequence of S consisting of those elements of S which are $\geqq a_i$ and $<a_{i+1}$; let T_m be the subsequence of S consisting of those elements of S which are $\geqq a_m$. Denote by t_i the cardinality of T_i. Then S is the disjoint union of the T_i and $\sum_{i=1}^m t_i=n$. Now

$$\sum_{i=1}^{m}(t_i+m-i)=n+\frac{m(m-1)}{2}$$

or

$$\frac{1}{m}\sum_{i=1}^{m}(t_i+m-i)=\frac{n}{m}+\frac{m}{2}-\frac{1}{2}.$$

Minimizing over m shows that $(1/m)\sum_{i=1}^m(t_i+m-i)\geqq(2n)^{1/2}-\frac{1}{2}$. Thus there is an index j such that $t_j+m-j\geqq(2n)^{1/2}-\frac{1}{2}$. Define $k=j+1+[\log_2 j]$. (Here and throughout $[x]$ denotes the least integer $\geqq x$). Put $V=\{a_k, a_{k+1},\cdots, a_m\}$; V is a sum-free sequence because its elements grow so rapidly that each element of V is greater than the sum of all smaller

Received by the editors April 23, 1973.
AMS (MOS) subject classifications (1970). Primary 10L10.
[1] The preparation of this paper was supported in part by NSF grant GP-3358X.

elements of V. The sequence T_j is sum-free because each element of T_j is $\geqq a_j$ and $<2a_j$, and as a result the sum of any two elements of T_j is greater than any element of T_j. Finally,

$$\begin{aligned}\text{the sum of all the elements of } T_j &< t_j \cdot 2a_j \\ &= 2^{(1+\log_2 t_j)}a_j \leqq 2^{1+\lceil \log_2 t_j \rceil}a_j \\ &\leqq a_{j+1+\lceil \log_2 t_j \rceil} = a_k \quad \text{if } k \leqq m.\end{aligned}$$

It is immediate that if $a_i \in V$, then a_i is greater than the sum of all smaller elements of $T_j \cup V$ and hence $T_j \cup V$ is a sum-free subsequence of S with

$$\begin{aligned}\text{number of elements} &\geqq t_j + m - k + 1 = t_j + m - j - [\log_2 j] \\ &\geqq (2n)^{1/2} - \tfrac{1}{2} - \log_2(2j) \geqq (2n)^{1/2} - \tfrac{1}{2} - \log_2(2n) \\ &\geqq (2n)^{1/2} - \log_2(4n).\end{aligned}$$

References

1. S. Choi, *The largest sum-free subsequence from a sequence of n numbers*, Proc. Amer. Math. Soc. **39** (1973), 42–44.

2. P. Erdös, *Extremal problems in number theory*, Proc. Sympos. Pure Math., vol. 8, Amer. Math. Soc., Providence, R.I., 1965, pp. 181–189. MR **30** #4740.

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