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MULTIPLE POINTS OF TRANSIENT RANDOM WALKS

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ABSTRACT. We determine the asymptotic behavior of the expected numbers of points visited exactly j times and at least j times in the first n steps of a transient random walk on a discrete Abelian group. We prove that the strong law of large numbers holds for these multiple point ranges.

Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables (taking values in an arbitrary countable Abelian group), and let $S_1=X_1$, $S_2=X_1+X_2$, $S_3=X_1+X_2+X_3+\cdots$ be the sequence of partial sums of $\{X_n\}$. The range R_n of the random walk associated with $X_1, X_2, \cdots$ is the number of distinct values assumed by the finite sequence of partial sums $S_1, S_2, \cdots, S_n$. Dvoretsky and Erdös [**1**] (and by different means Spitzer, Kesten and Whitman [**3**, pp. 38–40]) have shown that for any random walk $\lim_{n\to\infty} E(R_n/n)=1-F$, where F is the probability that $\exists n \ni S_n=0$, and, in fact, the strong law of large numbers applies to the range in the sense that $\lim_{n\to\infty} R_n/n=1-F$ a.s. (The latter result is of interest principally when the walk is transient, i.e. $F\neq 1$; Dvoretsky and Erdös obtain the stronger $\lim_{n\to\infty} R_n/E(R_n)=1$ a.s. for simple random walk in the plane.)

We obtain analogous results concerning the number of distinct values assumed exactly j times and at least j times by the sequence $S_1, S_2, \cdots, S_n$.

We introduce the notation:

R_n^j = the # of distinct values occurring at least j times in the sequence $S_1, \cdots, S_n$.
$R_n^{(j)}$ = the # of distinct values occurring exactly j times in the sequence $S_1, \cdots, S_n$.

Our main result is:

THEOREM. *For any random walk and any j, $\lim_{n\to\infty}(R_n^j/n)=(1-F)F^{j-1}$ a.s., and $\lim_{n\to\infty}(R_n^{(j)}/n)=(1-F)^2F^{j-1}$ a.s.*

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Note. (1) For recurrent walks this result is a direct consequence of the result for $j=1$, and of no interest.

(2) The corresponding result for simple walk on Z^n, $n\geqq 1$, is stated by Erdös and Taylor [2]. Their method is entirely different and applies only to walks on Z^n.

PROOF. The proof is inductive. The result is already known for R_n^1; so all we need do is establish the induction step: $R_n^j/n\to(1-F)F^{j-1}$ a.s. implies $R_n^{(j)}/n\to(1-F^2)F^{j-1}$ a.s. and $R_n^{j+1}/n\to(1-F)F^{j-1}$ a.s.

We assume then that $R_n^j/n\to(1-F)F^{j-1}$ a.s. and turn to the first order of business, the determination of the behavior of $E(R_n^{(j)})/n$ and $E(R_n^{j+1})/n$ as $n\to\infty$. To this end we write:

$$\begin{aligned} Z_k &= 1, \quad \text{if there are exactly } j-1 \text{ indices } l<k \ni S_l = S_k, \\ &= 0, \quad \text{otherwise.} \\ Y_{k,n} &= 1, \quad \text{if there are exactly } j-1 \text{ indices } l<k \text{ such that} \\ &\qquad\quad S_l = S_k \text{ and } S_{k+1}\neq S_k,\ S_{k+2}\neq S_k, \cdots, S_n\neq S_k, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

We have $R_n^{(j)}=\sum_{k=1}^n Y_{k,n}$ so that $E(R_n^{(j)})=\sum_{k=1}^n E(Y_{k,n})$. Also, $X_{k+1},\cdots,X_n$, are independent of Z_k, hence

$$\begin{aligned} E(Y_{k,n}) &= E(Z_k)P(X_{k+1}\neq 0, X_{k+1}+X_{k+2}\neq 0,\cdots, X_{k+1}+\cdots+X_{k+n}\neq 0) \\ &= E(Z_k)P(S_1\neq 0,\cdots,S_{n-k}\neq 0). \end{aligned}$$

But also $R_n^j=\sum_{k=1}^n Z_k$ and, in view of the induction hypothesis and the Lebesgue dominated convergence theorem,

$$\sum_{k=1}^{n} E(Z_k)/n = E(R_n^j)/n \to (1-F)F^{j-1}. \tag{1}$$

Furthermore, if we write, $f_k=P(S_1\neq 0, S_2\neq 0,\cdots, S_{k-1}\neq 0, S_k=0)$ we have $P(S_1\neq 0,\cdots,S_{n-k}\neq 0)=1-\sum_{i=1}^{n-k} f_i$ and, as $n\to\infty$,

$$1-\sum_{i=1}^{n-k} f_i \to 1-F. \tag{2}$$

A straightforward argument then yields

$$E(R_n^{(j)})/n \to (1-F)^2F^{j-1}, \tag{3}$$

and the further observation that $R_n^{j+1}=R_n^j-R_n^{(j)}$ gives

$$E(R_n^{j+1})/n \to (1-F)F^j. \tag{4}$$

To obtain the desired strong laws of large numbers we introduce the new sequences of random variables defined for $n=1, 2, \cdots, k=1, 2, \cdots$:

$T_{k,n}^{(j)}$ = the number of distinct values occurring exactly j times in $S_{(k-1)n+1}, \cdots, S_{kn}$,

$T_{k,n}^{j+1}$ = the number of distinct values occurring at least $j+1$ times in $S_{(k-1)n+1}, \cdots, S_{kn}$.

For any m we have

$$R_{mn}^{(j)} \leqq \sum_{k=1}^{m} T_{k,n}^{(j)} + E_{m,n}, \tag{5}$$

$$R_{mn}^{j+1} \leqq \sum_{k=1}^{m} T_{k,n}^{j+1} + E_{m,n}, \tag{6}$$

where $E_{m,n}$= the number of distinct sums which occur in at least two different blocks

$$\{S_{(k-1)n+1}, \cdots, S_{kn}\}, \qquad \{S_{(k'-1)n+1}, \cdots, S_{k'n}\}, \qquad k, k' \leqq m.$$

We prove below that

$$\limsup_{n\to\infty} \limsup_{m\to\infty} E_{m,n}/mn = 0,$$

and complete our proof of the theorem assuming the truth of this lemma.

We let $m\to\infty$ in (5) and obtain

$$\limsup_{m\to\infty} \frac{R_{mn}^{(j)}}{mn} \leqq \lim_{m\to\infty} \frac{\sum_{k=1}^{m} T_{k,n}^{(j)}}{mn} + \lim_{m\to\infty} \frac{E_{m,n}}{mn}. \tag{7}$$

By the strong law of large numbers applied to the sequence of (bounded) independent identically distributed random variables $T_{k,n}$, $k=1, 2, \cdots$,

$$\lim_{m\to\infty} \frac{\sum_{k=1}^{m} T_{k,n}^{(j)}}{mn} = \frac{1}{n} \lim_{m\to\infty} \frac{\sum_{k=1}^{m} T_{k,n}^{(j)}}{m} = \frac{E(T_{1,n}^{(j)})}{n} = \frac{E(R_n^{(j)})}{n}. \tag{8}$$

(7) yields a bound on $\limsup R_k^{(j)}/k$ as $k\to\infty$ through a subsequence consisting of multiples of n. The restriction of the mode of approach to a subsequence may be eliminated by observing that

$$\begin{aligned} R_k^{(j)}/k &\leqq \frac{R^{(j)}_{[k/n]_n+(k-[k/n]_n)}}{[k/n]_n} \leqq R^{(j)}_{[k/n]_n}/[k/n]_n + 1/[k/n] \\ &= R_{mn}^{(j)}/mn + o(1), \end{aligned}$$

so that $\limsup_{k\to\infty} R_k^{(j)}/k \leqq \limsup_{m\to\infty} R_{mn}^{(j)}/mn$ for any n.

Substituting (8) in (7), letting $n \to \infty$ and using (3) yields

$$\limsup_{k\to\infty} R_k^{(j)}/k \leqq (1-F)^2 F^{j-1} \quad \text{a.s.} \tag{9}$$

Trivial adjustments in the argument which led from (5) to (9) allow us to use (6) to conclude

$$\limsup_{k\to\infty} R_k^{j+1}/k \leqq (1-F)F^j. \tag{10}$$

But now we complete our argument with ease. For,

$$\liminf_{k\to\infty} R_k^{(j)}/k \geqq \liminf_{k\to\infty} R_k^j/k - \limsup R_k^{j+1}/k,$$

and in view of our induction hypothesis and (10) the right side is $\geqq (1-F)^2F^{j-1}$. Finally

$$\lim_{k\to\infty} R_k^{j+1}/k = \lim_{k\to\infty} R_k^j/k - \lim_{k\to\infty} R_k^{(j)}/k = (1-F)F^j.$$

Proof of Lemma. We define the random variable ξ_i^n by

$$\begin{aligned} \xi_i^n &= 1, \quad \text{if } S_i \neq S_{i+1}, S_i \neq S_{i+2}, \cdots, S_i \neq S_{([i/n]+1)_n} \\ &\qquad \text{but } \exists k > i \ni S_k = S_i \text{ or } n|i \text{ and } \exists k > i \ni S_k = S_i, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Clearly $0 \leqq E_{m,n} \leqq \sum_{i=1}^{mn} \xi_i^n$ (in fact, $E_{m,n} \leqq \sum_{i=1}^{(m-1)n} \xi_i^n$). We now choose an n' corresponding to n so that $n' \to \infty$ as $n \to \infty$ and $n'/n \to 0$. We let

$$\begin{aligned} \eta_i^n &= 1, \quad \text{if } S_i \neq S_{i+1}, S_i \neq S_{i+2}, \cdots, S_i \neq S_{i+n'} \\ &\qquad \text{but } \exists k > i + n' \ni S_k = S_i, \\ &= 0, \text{ otherwise.} \end{aligned}$$

The sequence of random variables η_i^n is stationary ergodic, hence

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \eta_i^n = E(\eta_i^n) = \sum_{n'}^{\infty} f_j,$$

by the ergodic theorem. Also $\xi_i^n \leqq \eta_i^n$ unless $n|i$ or $[i/n]+n-i<n'$, so

$$0 \leqq E_{m,n} \leqq \sum_{i=1}^{mn} \xi_i^n + m \cdot n'.$$

Hence

$$0 \leqq \limsup_{m\to\infty} \frac{E_{m,n}}{mn} \leqq \lim_{m\to\infty} \frac{\sum_{i=1}^{mn} \xi_i^n}{mn} + \frac{n'}{n} = \sum_{n'}^{\infty} f_j + \frac{n'}{n}.$$

And as $n \to \infty$, $\sum_{n'}^{\infty} f_j + n'/n \to 0$ by our choice of n'.

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