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REMARKS ON THE CLASSIFICATION PROBLEM FOR INFINITE-DIMENSIONAL HILBERT LATTICES

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ABSTRACT. A lattice satisfying the properties of a Hilbert lattice, but possibly reducible, possesses the relative center property. The division ring with involution $(D, *)$, which coordinatizes a Hilbert lattice satisfying the angle-bisection axiom and having infinite dimension, is formally real with respect to the involution, in particular having characteristic zero. Also D has the property that finite sums of elements of the form $\alpha\alpha^*$ are of the form $\beta\beta^*$ for some $\beta \in D$.

1. **Introduction.** In this paper, we continue the project embarked upon in our papers [6], [7], [8] and [9]. This project is a study of lattices possessing the properties of being complete, orthocomplemented, atomistic, irreducible, separable and infinite dimensional, M-symmetric, and orthomodular. In [8], we christened such a lattice an infinite-dimensional Hilbert lattice. There are only three known examples of infinite-dimensional Hilbert lattices, namely the lattices of all closed subspaces of real, complex, and quaterionic Hilbert space. Our main question is whether there are, in fact, any others. Our present results show that an arbitrary infinite-dimensional Hilbert lattice shares certain properties of the three canonical lattices. Thus these results are consistent with the possibility of the answer "no" to our main question. In §2, we prove that any Hilbert lattice (minus the assumption of irreducibility) has the relative center property. Our proof uses a result of M. Janowitz. §3 employs the concept of angle-bisection of orthogonal atoms, introduced in [9]. Here we show that, if our lattice L satisfies the angle-bisection axiom, then the division ring with involution coordinatizing L has several algebraic properties which bring it more closely into resemblance with the three canonical division rings.

2. **The relative center property.** It is well known that any interval sublattice $L(0, x)=\{y \in L: 0\leqq y\leqq x\}$, L a Hilbert lattice, $x \in L$, is complete, orthomodular, atomistic, M-symmetric, and separable. We show,

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in this section, that $L(0, x)$ is also irreducible and that $L(0, x)$ is a Hilbert lattice. We do this by means of the following result:

2.1. THEOREM (THE RELATIVE CENTER PROPERTY). *Let L be a lattice possessing all the properties of a Hilbert lattice, except possibly irreducibility. Then, for any $x \in L$, the center of $L(0, x)$ consists precisely of the set $\{z \wedge x : z$ is in the center of $L\}$.*

Before giving the proof, we remark that this property is one of those specified by S. S. Holland, Jr. [**1**, Theorem 4] as being valid in each of the three canonical lattices, but failing for an arbitrary complete orthomodular lattice. For x a finite element of L, the result is contained in Ramsay [**10**]. In fact, the referee has pointed out that, using the terminology of [**10**] and modifications of the proofs of some results of that paper, one can derive the relative center property in any semifinite dimension lattice. Our proof makes use of a result of M. Janowitz. In any complete orthomodular lattice, we say $a\ S\ b$ in case $a \wedge b = 0$ and $(a \vee b) \wedge y = (a \wedge y) \vee (b \wedge y)$ for all $y \in L$, and $a \nabla b$ in case $(a \vee x) \wedge b = x \wedge b$ for each $x \in L$. It is well known that ∇ implies S in general, but Janowitz has proved [**3**, 4.4] that, in a complete orthomodular lattice L, S implies ∇ if and only if L has the relative center property. The arguments which follow show that S implies ∇ in any (possibly reducible) Hilbert lattice. For the remainder of this section, we assume that L has all the properties of a Hilbert lattice, except possibly irreducibility.

2.2. LEMMA. *Suppose $a, b \in L$ with $a\ S\ b$, suppose that p, q are atoms in L with $p \leqq a$ and $q \leqq b$. Then, given $y \in L$ with $p \nleqq y$ and $q \nleqq y$, we can conclude $p \nleqq q \vee y$.*

PROOF. Suppose that there exists $y \in L$ such that $p \nleqq y$, $q \nleqq y$, and $p \leqq q \vee y$. Since $p \nleqq y$, we conclude, by the atomic exchange property [**5**, Definition 7.8], that $q \leqq p \vee y$. Thus $p \vee y = q \vee y$ and $p \wedge y = q \wedge y = 0$, so that p and q are perspective [**11**, Lemma 3.2] and thus strongly perspective (since L is finite modular, by [**5**, Lemma 27.9], we can apply the proof of Lemma 3.1 of [**11**] to get this result). Thus, there exists an atom r such that $p \vee r = q \vee r = p \vee q$ and $p \wedge r = q \wedge r = 0$. Now $r \leqq p \vee q \leqq a \vee b$ so that $r = (a \vee b) \wedge r$ which, by hypothesis, equals $(a \wedge r) \vee (b \wedge r)$. Note that either $a \wedge r$ or $b \wedge r$ is 0, otherwise $r \leqq a \wedge b = 0$. Also, not both $a \wedge r$ and $b \wedge r$ are 0, otherwise $r = (a \wedge r) \vee (b \wedge r) = 0 \vee 0 = 0$. Hence, $a \wedge r$ or $b \wedge r$ is 0, but not both; or, putting it differently, either $r \leqq a$ or $r \leqq b$, but not both. In the first case, if $r \leqq a$, then since $p \leqq a$, we have $p \vee r \leqq a$. But then $p \vee q \leqq a$ so $q \leqq a$, a contradiction, since we would then have $q \leqq a \wedge b = 0$. In the other case, if $r \leqq b$, then $q \leqq b$ implies that $q \vee r = b$, so $p \vee q \leqq b$, so $p \leqq b$, again a contradiction.

2.3. LEMMA. *Suppose $a, b \in L$ with $a\,S\,b$. Let p, q be atoms in L with $p \leqq a$ and $q \leqq b$. Then $p\,S\,q$.*

PROOF. Let $y \in L$ be arbitrary. There are two cases:

Case 1. If either $p \leqq y$ or $q \leqq y$, then supposing for instance that $p \leqq y$, we would have that, since q and y form a modular pair (due to the finite modularity of L),

$$(p \vee q) \wedge y = p \vee (q \wedge y) = (p \wedge y) \vee (q \wedge y),$$

the desired conclusion.

Case 2. Suppose $p \nleqq y$ and $q \nleqq y$; so $p \wedge y = q \wedge y = 0$. Thus, our goal, in this case, is to prove $(p \vee q) \wedge y = 0$. But, by Lemma 2.2, $p \nleqq y$ and $q \nleqq y$ imply $p \nleqq q \vee y$. But then, by [5, Exercise 7.1], since $p \nleqq q \vee y$, we conclude $(q \vee p) \wedge y = q \wedge y = 0$, as desired.

PROOF OF THEOREM 2.1. Suppose that $a, b \in L$ with $a\,S\,b$. Let p, q be atoms with $p \leqq a$ and $q \leqq b$. We claim that $p\,\nabla\,q$ and thus, by [5, Theorem 11.7], that $a\,\nabla\,b$. By Lemma 2.3, we have that $p\,S\,q$. Expressed in the notation of von Neumann [11, p. 32] in which $D(a, b, c)$ symbolizes $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ and $D'(a, b, c)$ denotes $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, we have $D(p, q, y)$ for any $y \in L$. But since p and q are finite and L is finite modular, this implies $D'(q, y, p)$ which, for the same reason, forces $D(y, p, q)$. But clearly, this is equivalent to $D(p, y, q)$ which is precisely $(p \vee y) \wedge q = y \wedge q$, y arbitrary, which says $p\,\nabla\,q$, as desired.

3. **Angle-bisection in Hilbert lattices and properties of the coordinatizing division ring.** In [9], we devised a lattice theoretic version of the bisection of the angle between two orthogonal vectors in Hilbert space, by a vector lying in the plane they determine. We refer the reader to [9] for the rather complicated details of that definition, but will make one notational remark. If p and q are orthogonal atoms in a Hilbert lattice and r is a third atom with $r < p \vee q$, we write $r\,B\,(p, q)$ to denote that r bisects the angle between p and q. Throughout this section, we assume that L is an infinite-dimensional Hilbert lattice satisfying the angle-bisection axiom; that is, for any pair p, q of orthogonal atoms in L, there exists an atom $r \in L$ such that $r\,B\,(p, q)$. The results of this section specify properties of the division ring with involution $(D, {}^*)$ which coordinatizes L [5, Theorem 34.5], properties which are a consequence of the angle-bisection property and infinite dimensionality in L. In [9], we proved a theorem (2.10) which said, essentially, that if there exists a vector of a given "length" in one direction in V (V being the left vector space over D whose existence is guaranteed in [5, 34.5]), then there exists a vector of that same length in any direction. This property of $((D, {}^*), V, (\cdot, \cdot))$ is, in fact, equivalent to the validity of the angle-bisection axiom in L. In this section,

we will prove further that, under these same conditions, D is formally real with respect to $*$, hence having characteristic zero, and that the sum of finitely many elements of the form $\alpha\alpha^*$ in D has that same form.

Our ability to derive these results depends on the fact that, via angle-bisection, we can prove that there exists a vector of unit length in every direction. Before we can do this, we must know that there exists at least one unit vector in L, so we will make some remarks about "scaling the form." By [**5**, 34.5], there exists, for the given L, $((D, *), V, (\cdot, \cdot))$ such that L is ortho-isomorphic to the lattice $L_\perp((D, *), V, (\cdot, \cdot))$ of all $\perp$-closed subspaces of V, $\perp$ being the orthogonality relation induced by the conjugate-bilinear, Hermitian, nonisotropic form $(\cdot, \cdot)$. Now the involution $*$ on D and the form $(\cdot, \cdot)$ on V are not uniquely determined by L, in fact, for any *selfadjoint* element $\gamma \in D$, $\lambda^{*\overline{}} = \gamma^{-1}\lambda^*\gamma$ is an involutory anti-automorphism of D and $(x, y)^{\overline{}} = (x, y)\gamma$ is a conjugate-bilinear, Hermitian, nonisotropic form on V such that L is also ortho-isomorphic to $L((D, *^{\overline{}}), V, (\cdot, \cdot)^{\overline{}})$. We claim that there exists a vector of length 1 in V, for if not, letting e be an arbitrary nonzero vector in V, scale the form $(\cdot, \cdot)$ to $(\cdot, \cdot)^{\overline{}} = (\cdot, \cdot)(e, e)^{-1}$, noting that $(e, e)^{-1}$ is a selfadjoint element of D. Then $(e, e)^{\overline{}} = (e, e)(e, e)^{-1} = 1$. By [**9**, 2.10], if f is any vector in V orthogonal to e, there exists $\alpha \in D$ such that $(\alpha f, \alpha f)^{\overline{}} = (e, e)^{\overline{}} = 1$. From this, we conclude easily that there exists, in V, a vector of unit length in every direction.

3.1. THEOREM. *Let L be an infinite-dimensional Hilbert lattice satisfying the angle-bisection axiom.* (a) *Given n scalars, $\lambda_1, \cdots, \lambda_n$ in D with $\lambda_1\lambda_1^* + \cdots + \lambda_n\lambda_n^* = 0$, necessarily $\lambda_i = 0$ for all $i = 1, \cdots, n$. In particular, the characteristic of D is* 0. (b) *Given any $\lambda_1, \cdots, \lambda_n \in D$, there exists $\gamma \in D$ such that $\lambda_1\lambda_1^* + \cdots + \lambda_n\lambda_n^* = \gamma\gamma^*$.* (*In the case $* =$ identity, this says that the sum of squares is a square.*)

PROOF. By the above remarks, we can express V as the countably infinite join of orthogonal atoms $\{p_1, p_2, \cdots\}$ where $p_i = De_i$ and $(e_i, e_i)^{\overline{}} = 1$ for all $i = 1, 2, \cdots$.

(a) Given any $\lambda_1, \cdots, \lambda_n$ in D with $\lambda_1\lambda_1^* + \cdots + \lambda_n\lambda_n^* = 0$, then

$$(\lambda_1 e_1 + \cdots + \lambda_n e_n, \lambda_1 e_1 + \cdots + \lambda_n e_n)^{\overline{}} = 0$$

so that $\lambda_1 e_1 + \cdots + \lambda_n e_n$ is the zero vector (since $(\cdot, \cdot)^{\overline{}}$ is nonisotropic), so that $\lambda_i = 0$ for all $i = 1, \cdots, n$.

(b) Given any $\lambda_1, \cdots, \lambda_n \in D$, we can find, by [**9**, 2.10], $\gamma \in D$ such that

$$\begin{aligned}\lambda_1\lambda_1^* + \cdots + \lambda_n\lambda_n^* &= (\lambda_1 e_1 + \cdots + \lambda_n e_n, \lambda_1 e_1 + \cdots + \lambda_n e_n)^{\overline{}} \\ &= \gamma(e_{n+1}, e_{n+1})^{\overline{}}\gamma^* = \gamma\gamma^*,\end{aligned}$$

as desired.

4. **Concluding remarks.** There is a related ring-theoretic problem, which has been formulated by S. S. Holland, Jr. [**2**]. Holland's results, for the ring case, have suggested further properties of $(D, *)$ which one might hope to derive in the setting of an infinite-dimensional Hilbert lattice satisfying the angle-bisection axiom. Two important questions are whether, under the assumptions of §3, it is true that (i) for any $\alpha \in D$, there exists a selfadjoint $\beta \in D$ such that $\beta^2=\alpha\alpha^*$ and β doubly commutes with $\alpha\alpha^*$, and (ii) every selfadjoint element in D is central?

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