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ON MODULAR LATTICES OF ORDER DIMENSION TWO

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ABSTRACT. In this note, it is shown that a modular lattice has order dimension $\leqq 2$ if and only if it contains no subset isomorphic to one of five described partially ordered sets.

The order dimension of a partially ordered set $(S, \leqq)$ is defined as the smallest cardinal number m such that the relation $\leqq$ is the intersection of m (linear) orders on S (Dushnik and Miller [3]). For the order dimension we have the following compactness theorem (Harzheim [4] and also the review of K. A. Baker, MR **43** #113):

THEOREM 0. *Let $(S, \leqq)$ be a partially ordered set, and let n be a natural number. If every finite subset of S has order dimension $\leqq n$, then $(S, \leqq)$ also has order dimension $\leqq n$.*

In Baker, Fishburn and Roberts [2], it is shown that for $n \geqq 2$ there is no finite list of partially ordered sets with the property: A partially ordered set $(S, \leqq)$ has order dimension $\leqq n$ if and only if no subset of S is isomorphic to one of the partially ordered sets in the list. Moreover, there does not exist such a list for lattices. The principal result of this note is that for modular lattices and $n=2$ we have a checking list with five finite partially ordered sets. Using Dilworth's theorem, Baker has proved in [1] that a (finite) distributive lattice has order dimension $\leqq n$ if and only if it does not contain the partially ordered set of atoms and coatoms of a boolean lattice with 2^{n+1} elements. For the proof of the principal result we need this theorem in the case $n=2$, hence we give a direct proof for this case without using Dilworth's theorem.

If a partial order $\leqq$ on S is the intersection of orders C_i, we say that $(S, \leqq)$ is *represented* by the chains (S, C_i); if $S := \{s_1, \cdots, s_n\}$, we describe a chain (S, C) by the sequence $s_{i_1}s_{i_2}\cdots s_{i_n}$ where $(s_{i_j}, s_{i_k}) \in C$ if and only if $j \leqq k$.

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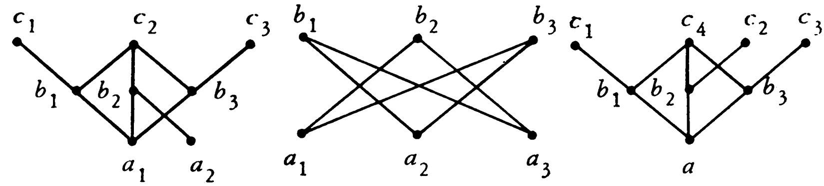
LEMMA 1. *The following partially ordered sets have order dimension* 3:



D_1^2 $\quad$ B_3 $\quad$ D^3

PROOF. The following can be easily checked:
B_3 is represented by

$$a_1a_2b_3a_3b_1b_2, \quad a_2a_3b_1a_1b_2b_3 \quad \text{and} \quad a_3a_1b_2a_2b_3b_1,$$

D^3 is represented by

$$ab_1c_1b_2c_2b_3c_3c_4, \quad ab_3c_3b_2c_2b_1c_1c_4 \quad \text{and} \quad ab_1b_2b_3c_4c_1c_2c_3,$$

D_1^2 is represented by

$$a_1b_1c_1b_3c_3a_2b_2c_2, \quad a_2b_2a_1b_3c_3b_1c_1c_2 \quad \text{and} \quad a_1a_2b_1b_2b_3c_2c_1c_3.$$

Now, we have to prove that there is no representation of B_3, D^3 and D_1^2 by any two chains. For B_3 this follows from the fact that, for every chain of a representation of B_3, there is at most one $i \in \{1, 2, 3\}$ with $b_i \cdots a_i$ in that chain. Suppose D^3 can be represented by two chains. Then, because of $b_m \nleq c_n$ for $m \neq n \neq 4$, those two chains have to be of the form

$$\cdots b_i \cdots c_i \cdots b_j \cdots c_j \cdots b_k \cdots c_k \cdots$$

and

$$\cdots b_k \cdots c_k \cdots b_j \cdots c_j \cdots b_i \cdots c_i \cdots \qquad (\{i, j, k\} = \{1, 2, 3\}),$$

which implies that $c_j \cdots c_4$ is in both chains; this contradicts $c_j \nleq c_4$ in D^3. Therefore, D^3 cannot be represented by two chains. Suppose D_1^2 can be represented by two chains. Then, because of $a_2 \nleq c_1$ and $a_2 \nleq c_3$, those two chains have to be of the form $a_1 \cdots c_i \cdots c_j a_2 \cdots c_2$ and

$$a_2a_1 \cdots c_2c_jc_i \qquad (\{i, j\} = \{1, 3\}),$$

which implies that $b_i \cdots c_j$ is in both chains; this contradicts $b_i \nleq c_j$ in D_1^2. Therefore, D_1^2 cannot be represented by two chains.

If D_3 is the dual of D^3 and D_2^1 the dual of D_1^2, Lemma 1 also shows that D_3 and D_2^1 have order dimension 3. For Lemma 2 we need the following notion: An element c of a lattice L with 0 is called a *chain element* of L if the interval $[0, c]$ is a chain.

LEMMA 2. *Let L be a finite distributive lattice having no subset isomorphic to B_3, and let c be a maximal chain element of L. Then $L\backslash[0, c]$ is a sublattice of L.*

PROOF. Obviously, the join of any two elements of $L\backslash[0, c]$ is again in $L\backslash[0, c]$. Suppose there are $a, b \in L\backslash[0, c]$ with $a\wedge b \in [0, c]$. Since c is a maximal chain element of L, there exist $a_1 \leqq a$ and $b_1 \leqq b$ with $a_1 \ngeqq c \nleqq b_1$ and $a_1 \nleqq c \ngeqq b_1$. Obviously, $a_1\wedge b_1 \in [0, c]$. We can assume, without loss of generality, that $a_1\wedge c \geqq b_1\wedge c$. Let a_2 and c_2 be covers of $a_1\wedge c$ with $a_2 \leqq a_1$ and $c_2 \leqq c$, and let b_2 be a cover of $b_1\wedge c$ with $b_2 \leqq b_1$. Then $b_2\vee(a_1\wedge c)$ covers $a_1\wedge c$; furthermore, $a_2 \neq b_2\vee(a_1\wedge c) \neq c_2 \neq a_2$. By distributivity, it follows that $\{a_2, b_2\vee(a_1\wedge c), c_2, b_2\vee c_2, a_2\vee c_2, a_2\vee b_2\}$ is a subset of L isomorphic to B_3. This is a contradiction to our assumption. Therefore, the meet of any two elements of $L\backslash[0, c]$ is again in $L\backslash[0, c]$, too.

THEOREM 3. *A distributive lattice has order dimension $\leqq 2$ if and only if it does not contain a subset isomorphic to B_3.*

PROOF. By Lemma 1, a distributive lattice of order dimension $\leqq 2$ cannot contain a subset isomorphic to B_3. Now, let L be a distributive lattice having no subset isomorphic to B_3. First we prove by induction on the cardinality of L: If L is finite and if c is any chain element of L, then L can be represented by two chains $a_1a_2\cdots a_n$ and $b_1b_2\cdots b_n$ such that $[0, c]=\{a_1, a_2, \cdots, a_i\}$ for some $i \leqq n$. Let d be a maximal chain element of L with $c \leqq d$. By Lemma 2, $L\backslash[0, d]$ is a sublattice of L, which has a least element v. Since L does not contain a subset isomorphic to B_3, $d\vee v$ is a chain element of $L\backslash[0, d]$. By the induction hypothesis, $L\backslash[0, d]$ can be represented by two chains $a_1a_2\cdots a_m$ and $b_1b_2\cdots b_m$ with $[v, d\vee v]=\{a_1a_2, \cdots, a_j\}$ for some $j \leqq m$. Now, it can be easily checked that L is represented by the two chains

$$0\cdots(d\wedge a_1)(d\wedge a_2)\cdots(d\wedge a_j)a_1a_2\cdots a_m$$

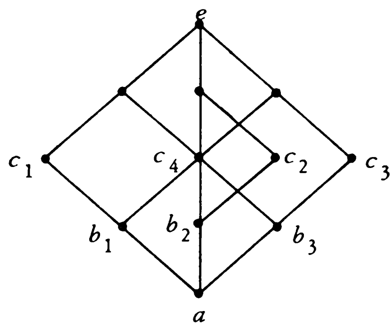
and

$$0\cdots(d\wedge a_1)b_{k_1}\cdots b_{k_2-1}(d\wedge a_2)b_{k_2}\cdots b_{k_j-1}(d\wedge a_j)b_{k_j}\cdots b_m$$

with $a_i=b_{k_i}$ for $1 \leqq i \leqq j$. This proves that L has order dimension $\leqq 2$ if L is finite. Now, let L be infinite. Since every finite subset of L generates a finite sublattice of L, each finite subset of L has order dimension $\leqq 2$. But then the whole lattice L also has order dimension $\leqq 2$ by Theorem 0.

LEMMA 4. *Let M be a modular lattice having no subset isomorphic to D^3. If $\{a, c_1, c_2, c_3, e\}$ is a five-element, nondistributive sublattice of M with $a<c_i<e$, then a is covered by c_i and e covers c_i $(i=1, 2, 3)$.*

PROOF. Suppose a is not covered by c_i or e does not cover c_i for some i. Then, by modularity, there is an element b_1 in M with $a<b_1<c_1$. Define $b_2 := (b_1 \vee c_3)\wedge c_2$, $b_3 := (b_1 \vee c_2)\wedge c_3$ and $c_4 := (b_1\vee c_2)\wedge(b_1 \vee c_3)$. Since $a<b_i<c_i$ and $c_i\wedge c_4=b_i$ for $1\leqq i\leqq 3$, the subset $\{a, b_1, b_2, b_3, c_1, c_2, c_3, c_4\}$ is isomorphic to D^3.



An element a of a lattice L is said to be *irreducible* if $b_1, c_1<a$ implies $b_1\vee c_1<a$ and if $a<b_2, c_2$ implies $a<b_2\wedge c_2$. Thus, irreducible elements are exactly those elements which can be deleted from a sublattice with the end result that a sublattice remains. Especially, $\mathfrak{N}(L) := \{b \in L \mid b$ is not irreducible$\}$ is a sublattice of L.

THEOREM 5. *For a modular lattice M the following conditions are equivalent*:

(i) *M has order dimension $\leqq 2$.*

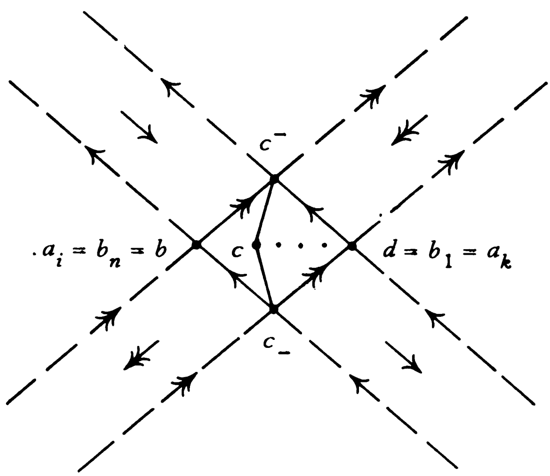
(ii) *$\mathfrak{N}(M)$ is distributive and has order dimension $\leqq 2$.*

(iii) *M does not contain a subset isomorphic to B_3, D^3, D_3, D_1^2 or D_2^1.*

PROOF. Since the order dimension of any subset of M cannot exceed the order dimension of M, (i)$\Rightarrow$(iii) is a consequence of Lemma 1.

(iii)$\Rightarrow$(ii): Let $\{a, b_1, b_2, b_3, c\}$ a five-element, nondistributive sublattice of M. By Lemma 4, a is covered by b_i and c covers b_i $(i=1, 2, 3)$. Since no subset of M is isomorphic to D^3, D_3, D_1^2 or D_2^1, it follows that at least one of the b_i's has to be irreducible. Therefore, $\{a, b_1, b_2, b_3, c\}$ is not contained in $\mathfrak{N}(M)$. Thus $\mathfrak{N}(M)$ is distributive and has order dimension $\leqq 2$ by Theorem 3.

(ii)$\Rightarrow$(i): Since $\mathfrak{N}(M)$ is distributive, every five-element, nondistributive sublattice M_3 of M must contain an irreducible element c, and we get by the same construction as in the proof of Lemma 4 that c has a cover c^{-} and a subcover c_{-} in M with $c^{-}, c_{-} \in M_3$. Let S be a sublattice of M generated by a finite subset F, and let I be the set of all irreducible elements of F contained in a five-element, nondistributive sublattice of M. We define $I^{-} := \{c^{-} \mid c \in I\}$ and $I_{-} := \{c_{-} \mid c \in I\}$. Since $(F\backslash I)\cup I^{-} \cup I_{-}$ is contained in the distributive sublattice $\mathfrak{N}(L)$, the finite set $(F\setminus I)\cup I^-\cup I_-$ generates a finite sublattice T of M. Furthermore, we have $a\vee c=a\vee c^-$ and $b\wedge c=b\wedge c_-$ for $a, b\in M$ and $c\in I$ if $a\vee c\neq c\neq b\wedge c$. Hence $S\subseteq T\cup I$, and therefore S is finite. Thus, every finitely generated sublattice of M is finite. Therefore, by Theorem 0, we can assume that M is finite. Let D be a maximal distributive sublattice of M with $\mathfrak{N}(M)\subseteq D$. By Theorem 3 and (ii), D has order dimension $\leqq 2$. Now, we prove for every subset $E\supseteq D$ by induction on the cardinality of E that E has order dimension $\leqq 2$. Suppose we have already seen that E can be represented by two chains $a_1a_2\cdots a_r$ and $b_1b_2\cdots b_r$. If there is a $c\in M$ with $c\notin E$, c is an irreducible element contained in a five-element, nondistributive sublattice. Since D is a maximal distributive sublattice containing $\mathfrak{N}(L)$ and since the length of the interval $[c_-, c^-]$ is two, there are exactly four elements c_-, b, d, c^- in the intersection $D\cap[c_-, c^-]$. There is, without loss of generality, $i<k$ with $a_i=b$ and $a_k=d$. Let j be the smallest number with $i<j$ and $a_i\nleqq a_j$. Furthermore, there is $l<n$ with $b_l=d$ and $b_n=b$. Let m be the greatest number with $m<n$ and $b_m\nleqq b_n$.



Now, we assert that $E\cup\{c\}$ is represented by $a_1\cdots a_{j-1}ca_j\cdots a_r$ and $b_1\cdots b_mcb_{m+1}\cdots b_r$. Since $x<c$ and $c<y$ implies $x\leqq c_-$ and $c^-\leqq y$, the two orders extend the partial order on $E\cup\{c\}$. Next we have to prove that $a_p=b_q$ with $j\leqq p$ and $m<q$ implies $c<a_p=b_q(c^-\leqq a_p=b_q)$. Suppose $c^-\nleqq a_p=b_q$. Then we have $a_i\nleqq a_p$ or $b_l\nleqq b_q$. If $a_i\nleqq a_p$, by $i<p$, it follows $q<n$. Because of $m<q$, we get $a_p=b_q\leqq b_n=a_i$ which contradicts $i<p$. If $a_i\leqq a_p$, we must have $b_l\nleqq b_q$. This forces $p<k$ because of $l<q$. Since $a_i\leqq a_p$, we have $b_n\leqq b_q$ and hence $n\leqq q$. If we let $c^-=b_s=a_t$ and $a_j=b_w$, then $k<t$, $n<s$, and $w<n$. Thus $w<q$, $w<s$, and $j<t$ because $p<k$.

Hence $c^- \nleqq a_p$, $a_i \leqq a_p$ and $a_i \nleqq a_j$ implies $a_j \leqq a_p$ and $a_j \leqq c^-$. Since c^- covers a_i, we get $a_j \leqq a_p \wedge c^- = a_i$ which contradicts $i < j$. Thus, $c^- \leqq a_p = b_q$ is proved. Dually, it follows that $a_q = b_p$ with $q < j$ and $p \leqq m$ implies $a_q = b_p < c$ $(a_q = b_p \leqq c_-)$. This finishes the proof that $E \cup \{c\}$ is represented by $a_1 \cdots a_{j-1} c a_j \cdots a_r$ and $b_1 \cdots b_m c b_{m+1} \cdots b_r$. Therefore, M has order dimension $\leqq 2$.

COROLLARY 6. *Every finitely generated modular lattice of order dimension $\leqq 2$ is finite.*

PROBLEM. *For which natural numbers $n \geqq 3$ does there exist a finite list of partially ordered sets $(S_i, \leqq_i)$ such that a modular lattice has order dimension $\leqq n$ if and only if it contains no subset isomorphic to some of the $(S_i, \leqq_i)$.*

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