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FULLY IDEMPOTENT RINGS HAVE REGULAR CENTROIDS

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ABSTRACT. We prove that the centroid of a ring all of whose ideals are idempotent is commutative and regular in the sense of von Neumann. The center of a fully idempotent ring is regular. Evidently every regular ring is fully idempotent. One nonregular example is Sasiada's simple radical ring. A subring of the countably infinite row-finite matrices over Sasiada's ring provides an example of a nonsimple, indecomposable, nonregular fully idempotent ring.

A ring is called fully idempotent when each ideal equals its square; this is equivalent to saying that each factor ring of the ring is semiprime [**1**, p. 418]. One result to be proved states that the centroid of a fully idempotent ring is a field if and only if the ring is indecomposable. Thus we deduce the following theorem:

THEOREM. *If each factor ring of a prime ring is semiprime, its centroid is a field.*

Our results could be applied to commutative rings; consequently we find it appropriate to prove, using statements in the literature, that fully idempotent duo rings are regular.

We will need the following theorem and corollary:

THEOREM A. *A ring R is fully idempotent if and only if $t \in HtH$ for every ideal H and every $t \in H$.*

PROOF OF NECESSITY. If $t \in Y$ where Y is a fully idempotent ring, then $t \in YtY$ [**1**, Proposition 2.3, p. 421]. It is clear, however, that an ideal H of a fully idempotent ring R has only idempotent ideals. For let V be an H-ideal and let $\bar{V}=V+VR+RV+RVR$. Then $\bar{V}\subseteq H$ so that $\bar{V}^3\subseteq HVH\subseteq V$. Thus the R-ideal $\bar{V}=\bar{V}^3\subseteq V$, so that $V=\bar{V}$ is idempotent.

COROLLARY. *$H=HtH=RtR$, when H is the ideal generated by t.*

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1. **The centroid of a fully idempotent ring.** Rings are not assumed to have a unit element. We shall denote by $C(R)$ or C the centroid of a ring R (please see [3, p. 64] for the definition). Evidently, when R has a unit element, the center of R is its centroid.

Notation. We shall write $x \to x^c$, $x \in R$, $c \in C$.

We have from [3, p. 64] that

(*) $(xy)^c = x^c y = xy^c$, $x, y \in R$, $c \in C$,

(**) C is commutative if $R = R^2$.

LEMMA 1.1. *Let $c \in C$ and $t \in R$. Then the ideal RtR is invariant under c.*

PROOF. Apply (*) to the typical element $\sum r_i t s_i$.

In the results to follow the ring R is fully idempotent.

LEMMA 1.2. *Let $c \in C$, and let $t \in R^c$. Then there is an element $x \in RtR$ such that $x^c = t$.*

PROOF. If $z^c = t$, $(\sum r_i z s_i)^c = \sum r_i t s_i$ for all choices of r_i and s_i, whence $RtR \subseteq R^c$. By the corollary to Theorem A, $t = \sum_1^k m_i t n_i$ for some elements m_i and n_i in RtR. Thus for $i = 1, \cdots, k$, elements p_i exist such that $p_i^c = m_i$. Let $x = \sum p_i t n_i$. Then $x^c = \sum m_i t n_i = t$, as required.

LEMMA 1.3. *If $c \in C$, then $R^c \cap (\ker c) = 0$.*

PROOF. Let $t \in (\ker c) \cap R^c$. For some $x \in RtR$, $x^c = t$ by Lemma 1.2. But $(RtR)^c = Rt^cR = 0$, so that $t = x^c = 0$.

For any element $r \in R$, we have shown that $r^c = x^c$ for some $x \in Rr^cR$. Thus $r = x + (r - x)$ belongs to $R^c + (\ker c)$. Considering Lemma 1.3, we have proved the following theorem:

THEOREM 1.4. $R = R^c \oplus (\ker c)$.

THEOREM 1.5. *The centroid C of a fully idempotent ring R is regular and commutative. It is a field if and only if R is not the direct sum of two nonzero ideals.*

PROOF. C is commutative by (**). In the indecomposable case, if $0 \neq c \in C$, $R^c = R$ and $(\ker c) = 0$ by Theorem 1.4, so that c is invertible; C is a field. To conclude the proof of the second statement we note that C has zero divisors when R is decomposable (the implied projections belong to C).

If $c \in C$ and $x \in R^c$, Lemma 1.2 implies that $x = y^c$ for some $y \in R^c$; this y is unique since $R^c \cap (\ker c) = 0$. We define a map k by

$$t^k = t \qquad (t \in (\ker c)),$$

$$x^k = y \qquad (x \in R^c),$$

where $y \in R^c$ is such that $y^c = x$. It is easily verified that $k \in C$ and that $c = ckc$. C is a commutative regular ring.

COROLLARY. *The center S of R is regular.*

PROOF. C contains S, since R is a faithful R-module and thus is a faithful S-module. For each $c \in C$, $s \in S$ and $r \in R$ we have

$$r^{sc} = (rs)^c = rs^c, \qquad s^c r = (sr)^c = (rs)^c = rs^c.$$

Thus the function sc belongs to S; S is an ideal of the commutative regular ring C, whence S is a regular ring [**3**, Theorem 22, p. 30].

A ring having no strictly one-sided ideals is called a duo ring.

THEOREM 1.6. *A fully idempotent duo ring is regular.*

PROOF. Lajos has proved [**4**, Theorem 2]: A ring R is a regular duo ring if and only if $A \cap B = AB$ for every left ideal A and every right ideal B. The equality holds for the ideals A and B of a fully idempotent ring (please see [**1**, (C) of Theorem 1.2, p. 418]).

2. **An example of an indecomposable, nonregular, fully idempotent ring.** Sasiada's simple radical ring [**5**] is an example, since the radical of a regular ring is zero [**2**, p. 42]. We present a nonsimple example.

Notation. Let S be a ring. Let $A = A(S)$ be the set of all row-finite matrices with entries in S, where the number of rows and columns is countably infinite. If $t \in S$, tE_{ij} will signify the matrix with t in the (i,j) position and zeros elsewhere. $B = B(S)$ will denote the subring of A generated by all matrices in the form tE_{ij}, $t \in S$, $1 \leqq i, j < \infty$; $Y(S)$ will be the subring of A all of whose elements have the form $\mathrm{diag}(s, \cdots)$ for some $s \in S$. Let $T = T(S)$ be the subring of A generated by $Y(S)$ and $B(S)$; thus $T(S) = Y(S) + B(S)$, a direct sum of groups.

REMARK 2.1. $B(S)$ is an ideal of $T(S)$.

REMARK 2.2. $B(S)$ is locally matrix over S. Thus $B(S)$ is simple if S is simple; $B(S)$ is a radical ring if S is a radical ring.

PROPOSITION 2.3. *If S is a simple ring, $T(S)$ is a fully idempotent ring with minimum nonzero ideal $B(S)$.*

PROOF. For $v \in T$ which does not belong to B, we write $v = \mathrm{diag}(s, \cdots) + b$ where $0 \neq s \in S$ and where, for some positive integer n, b is a matrix whose (i,j) entry is zero for $i, j > n$. Fix $m > n$. For each p and q in S we have

$$psqE_{ij} = (pE_{im})v(qE_{mj}) = (pE_{im})[\mathrm{diag}(s, \cdots)](qE_{mj})$$

in the ideal V generated by v. Thus the ideal V contains B and has in it the element diag($s, \cdots$) just described. Clearly, $Y(S) \subseteq V$; $V=T$; T and B are the only nonzero ideals of T. The simple ring B equals its square. Then $T=T^2$, since $Y(S)=[Y(S)]^2$.

THEOREM 2.4. *If S is a simple radical ring, $T(S)$ is a nonsimple, indecomposable, nonregular, fully idempotent ring.*

PROOF. By Proposition 2.3, T is nonsimple, indecomposable, and fully idempotent. Each element of B has a quasi-inverse by Remark 2.2. Since $T(S)/B(S) \simeq S$ has no primitive ideals, $T(S)$ is a radical ring and is nonregular.

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