ON THE INVARIANCE OF CERTAIN CLASSES OF FUNCTIONS UNDER COMPOSITION

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ABSTRACT. Certain classes of functions are mapped into themselves by any change of variable. For some classes of this type which are of interest in the study of Fourier series, it is shown that the necessary and sufficient condition that $g \circ f$ be in the class for each f of that class is that $g \in \text{Lip } 1$.

1. The class of functions of bounded variation on an interval has the property of invariance under a change of variable, i.e., if $f \in BV$ and h is a homeomorphism of the interval onto itself, then $f \circ h \in BV$. This is a property of several classes of interest in the study of Fourier series. Among these are the class GW of functions with only simple discontinuities which have a convergent Fourier series after every change of variable, and the class UGW defined analogously with respect to uniform convergence.

Another such class is the class HBV , the functions of harmonic bounded variation, defined as follows. If $I = [a, b]$, write $f(I) = f(b) - f(a)$. We say that $f \in HBV$ if $\sum_1^\infty f(I_n)/n < \infty$ for every collection $\{I_n\}$ of non-overlapping intervals.

It is known that ([3], [6]) $GW \supseteq HBV$, and that ([1], [6]) $GW_{(\text{cont.})} \supsetneq UGW \supseteq HBV_{(\text{cont.})}$.

Here we are concerned with the question of which functions have the property that they leave these classes invariant under composition on the left. We assume throughout that the domain of such a function g is an interval and $g \circ f$ is well defined. We have the following

THEOREM. $g \circ f$ is in one of the classes GW , UGW , or HBV for each f of that class whose range is in the domain of g if and only if $g \in \text{Lip } 1$.

2. The necessary and sufficient conditions for a function to belong to GW or UGW are phrased in terms of "systems of intervals" at a point. For each n , let I_{nm} , $m = 1, 2, \dots, k_n$, be disjoint closed intervals such

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that $I_{n,m-1}$ is to the left of I_{nm} . Let there be a real x such that for every $\delta > 0$ there is an N such that $I_{nm} \subset (x, x + \delta)$ whenever $n > N$. Then the collection $\mathfrak{I} = \{I_{nm} : n = 1, 2, \dots; m = 1, 2, \dots, k_n\}$ is called a right system of intervals at x . A left system is defined similarly. Goffman and Waterman [4] also required that $\lim_{n \rightarrow \infty} k_n = \infty$ and $\lim_{n \rightarrow \infty} n^{-1}k_n = 0$. Baernstein and Waterman [1] showed this requirement to be redundant.

Goffman and Waterman [4] showed that $f \in GW$ if and only if

$$(1) \quad \lim_{n \rightarrow \infty} \sum_{m=1}^{k_n} f(I_{nm})/m = 0$$

for every system $\mathfrak{I}$. Baernstein and Waterman [1] showed that a continuous $f \in UGW$ if and only if both (1) and

$$(2) \quad \lim_{n \rightarrow \infty} \sum_{m=1}^{k_n} f(I_{nm})/(k_n + m - 1) = 0$$

hold for every system $\mathfrak{I}$.

It is clear that if we add to (1) and (2) the requirement that for each n the sign of $f(I_{nm})$ be constant, we obtain equivalent conditions and, therefore, if we replace $f(I_{nm})$ in these conditions by $|f(I_{nm})|$, we obtain still another set of equivalent conditions.

Since $g \in \text{Lip } 1$ implies that there is a $C > 0$ such that $|g \circ f(I)| < C|f(I)|$ for every interval I , it is obvious, in view of the above remarks, that composition on the left with functions of class $\text{Lip } 1$ preserves GW , UGW , and HBV .

We will now show that continuity of g is a necessary condition for preservation of these classes. One way in which this might be done is to adapt the classical constructions of a function whose Fourier series diverges at a point to fit our situation. It is far simpler, however, to use the characterizations of these classes given above.

Without loss of generality, we may suppose that $g(0) = 0$ and $g(t_n) \geq 1$, where $t_n \searrow 0$ and $\sum_{n=1}^{\infty} t_n < \pi$. Let $\{I_n\}$ be a sequence of disjoint closed intervals in $(0, \pi)$ which converges downward to the origin. Define f to be t_n on I_n for each n and to be zero at $-\pi, 0, \pi$, and at the midpoint of the interval between I_n and I_{n+1} for each n . On each of the component intervals of the set in $[-\pi, \pi]$ on which f is not yet defined, let f be linear and such that the resulting function is continuous on $[-\pi, \pi]$. Clearly $g \circ f$ is not in the class GW and, therefore, not in either UGW or HBV . However, f is a continuous function of bounded variation and is, therefore, in all three of these classes.

3. We will now show that the condition on g is necessary. We require certain information for this demonstration.

If f is a continuous function on an interval, the Banach indicatrix of f , $n(y; f)$ is defined to be the cardinality of $\{x: f(x)=y\}$ if this set is finite and $+\infty$ otherwise. The class GS of functions for which $\int \log n(y; f) dy < \infty$ was introduced by Beurling. Integration here is over the range of f . It was shown by Garsia and Sawyer [2] that the functions of class GS have uniformly convergent Fourier series. Clearly $UGW \supseteq GS$. It is also true ([3], [6]) that $HBV_{(\text{cont.})} \supseteq GS$.

It is interesting to note that these inclusions are *proper*. This follows from a result of Kiselman [5] to the effect that GS is not closed under addition. Clearly the other classes are linear spaces and, indeed, HBV and $HBV_{(\text{cont.})}$ are Banach spaces [6].

We shall show that if $g \notin \text{Lip } 1$, then there is a function $f \in GS$ such that $g \circ f \notin GW$.

Since we may assume that g is continuous, it is easy to see that, without loss of generality, we may suppose that there is a sequence of disjoint closed intervals J_n in $(0, 1)$ with endpoints $\searrow 0$ and a sequence of real numbers $c_n \rightarrow \infty$ such that $\sum_1^\infty 1/c_n < \infty$ and $|J_n|c_n < g(J_n)$. Clearly $g(J_n) \rightarrow 0$. Let

$$k_n = [\exp(1/c_n |J_n|)] + 1.$$

Then $k_n \rightarrow \infty$ and

$$g(J_n) \log k_n \geq g(J_n)/c_n |J_n| \geq 1.$$

Also, for some $C > 0$,

$$\sum_1^\infty |J_n| \log k_n < C \sum_1^\infty 1/c_n < \infty.$$

For each $n=1, 2, \dots$, let $I_{n1}, I_{n2}, \dots, I_{nk_n}$ be a collection of disjoint closed intervals in $(\pi/2^n, \pi/2^{n-1})$ with I_{nm} to the left of $I_{n,m+1}$ for each m . Let f be defined on each I_{nm} , $m=1, \dots, k_n$; $n=1, 2, \dots$, to be the increasing linear map of I_{nm} onto J_n . Let $f(-\pi)=f(0)=f(\pi)=0$. Define f to be linear on each of the component intervals of the remainder of $[-\pi, \pi]$ and continuous on $[-\pi, \pi]$. Clearly

$$\int \log n(y; f) dy \leq \log 2 + \sum_1^\infty |J_n| \log 2k_n < \infty,$$

and so $f \in GS$. But

$$\sum_{m=1}^{k_n} g \circ f(I_{nm})/m = g(J_n) \sum_1^{k_n} 1/m > g(J_n) \log k_n \geq 1.$$

Hence $g \circ f \notin GW$.

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