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CLASS NUMBERS AND μ-INVARIANTS OF CYCLOTOMIC FIELDS

T. METSÄNKYLÄ

ABSTRACT. We give a new upper bound for the μ-invariant of a cyclotomic field by estimating the first factor of the class number of the pth cyclotomic field (p an odd prime).

For each $n \geqq 0$ let h_n denote the class number of the cyclotomic field of p^{n+1}th roots of unity, where p is an odd prime. According to Iwasawa [1], the greatest exponent $e(n)$ for which $p^{e(n)} | h_n$ is given by a formula

$$e(n) = \lambda n + \mu p^n + \nu,$$

valid for all sufficiently large n. Here λ, μ, and ν are integers ($\lambda, \mu \geqq 0$) independent of n. In [2] Iwasawa proved the following estimates for μ:

(i) $\mu < p-1$ for all p,

(ii) if $c > \frac{1}{2}$, then there exists a bound $N(c)$ such that $\mu < c(p-1)$ whenever $p > N(c)$.

We shall show that $\mu < (p-1)/2$ for all p.

Let us denote by h^- the so-called first factor of h_0. As shown in [2], the problem of estimating μ can be reduced to that of estimating h^- by means of the relation $p^{\mu/2} \leqq h^-$.

It is known that

$$h^- = (2p)^{-(p-3)/2} \left| \prod_{\chi \in S} \sum_{n=1}^{p-1} \chi(n) n \right|,$$

where S denotes the set of all odd residue class characters mod p. Noting that

$$\begin{aligned} \sum_{\chi \in S} \chi(m)\chi'(n) &= (p-1)/2 && \text{if } m \equiv n \pmod{p},\ (mn, p) = 1, \\ &= -(p-1)/2 && \text{if } m \equiv -n \pmod{p},\ (mn, p) = 1, \\ &= 0 && \text{otherwise,} \end{aligned}$$

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(χ' means the complex conjugate of χ), we first get

$$\sum_{\chi\in S}\left|\sum_{n=1}^{p-1}\chi(n)n\right|^2=((p-1)/2)\left(\sum_{n=1}^{p-1}n^2-\sum_{n=1}^{p-1}n(p-n)\right)=(p-2)(p-1)^2p/12.$$

Therefore, by the arithmetic-geometric mean inequality,

$$\prod_{\chi\in S}\left|\sum_{n=1}^{p-1}\chi(n)n\right|^{4/(p-1)}\leqq (p-2)(p-1)p/6<p^3/6.$$

This gives us the estimate

$$h^- < 2p(p/24)^{(p-1)/4} \tag{1}$$

Thus, if $p>3$, we see that $h^-<p^{(p-1)/4}$ and so $\mu<(p-1)/2$. This holds also for $p=3$, since then $h^-=1$. (As a matter of fact, we know that $\mu=0$ for all regular primes.)

It should be mentioned that the result (1) has been obtained earlier by Lepistö [3] and the author [4] by more complicated methods than that presented above.

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Department of Mathematics, University of Turku, SF-20500 Turku, Finland