PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 43, Number 2, April 1974

EXTENDING CONTINUOUS LINEAR FUNCTIONALS IN CONVERGENCE INDUCTIVE LIMIT SPACES

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ABSTRACT. Let E_n be an increasing sequence of locally convex linear topological spaces such that the dual E_n' of each has a Fréchet topology (not necessarily compatible with the dual system (E_n', E_n)) weaker than the Mackey topology. Let $E=\bigcup_{n=1}^{\infty} E_n$, F be a subspace of E and τ the inductive limit convergence structure on E. Conditions are given which insure that every τ-continuous linear functional on F has a τ-continuous linear extension to E. This result generalizes a theorem of C. Foias and G. Marinescu.

Let (E, τ) be a convergence vector space and f a continuous linear functional defined on a subspace F of E. There are relatively few situations where it is known whether f has a continuous linear extension to E. In [3] C. Foias and G. Marinescu showed that, in inductive limits of reflexive Banach spaces, sequentially continuous linear functionals defined on sequentially closed subspaces have continuous linear extensions to the whole space. In the terminology of [2] this theorem yields, "If (E, τ) is a convergence inductive limit of a sequence of reflexive Banach spaces, then every τ-continuous linear functional on a τ-closed subspace of E has a τ-continuous linear extension to E." It is the purpose of this note to adapt the method of proof used in [3] to prove a more general result (Theorem 1). M. De Wilde [1] has extended the Foias-Marinescu theorem in a different direction to inductive limits of Banach spaces in which the canonical injections are weakly compact, relieving the reflexivity condition. This line of thought has been carried over to an extension theorem in the context of spaces with "boundedness structures" (espaces bornologiques) by H. Hogbe-Nlend [4, p. 66].

THEOREM 1. *Let $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ be a sequence of locally convex linear topological spaces such that for each n*

(1) *the canonical injection $i_n: E_n \to E_{n+1}$ is continuous,*

(2) *there is a Fréchet topology T_n on E'_n (not necessarily compatible with the dual system (E'_n, E_n)) weaker than the Mackey topology $\tau(E'_n, E_n)$, and*

(3) *the transpose maps $i_n^*: E'_{n+1} \to E'_n$ are continuous for the topologies T_{n+1} and T_n.*

Received by the editors December 1, 1972 and, in revised form, July 12, 1973.
AMS (MOS) subject classifications (1970). Primary 46A05, 46A20.

Let F be a linear subspace of $E=\bigcup_{n=1}^{\infty} E_n$, such that for each n, F_n is closed in E_n, where $F_n=E_n\cap F$. If φ is a linear functional on F such that for each n, $\varphi_n=\varphi|_{F_n}$ is continuous, then there exists a linear functional $\hat{\varphi}$ on E such that $\hat{\varphi}|_F=\varphi$ and $\hat{\varphi}|_{E_n}$ is continuous for each n.

The proof requires the following lemmas.

LEMMA 1. *Let E and F be two locally convex linear topological spaces, and $J: E\to F$ a continuous linear mapping. Let M be a subspace of E and N a closed subspace of F such that $J^{-1}(N)\subseteq M$. Then $J^*[N^\circ]$ is $\sigma(E',E)$ dense in M°.*

PROOF. It suffices to show that $[J^*(N^\circ)]^{\circ\circ}\supseteq M^\circ$. This would follow if $M\supseteq [J^*(N^\circ)]^\circ$. But if $z\in[J^*(N^\circ)]^\circ$, then for all $y\in J^*(N^\circ)$, $\langle z,y\rangle=0$. That is, for all $x\in N^\circ$, $0=\langle z,J^*(x)\rangle=\langle J(z),x\rangle$, and hence $J(z)\in N^{\circ\circ}=N$. But, then, $z\in J^{-1}(N)\subseteq M$.

LEMMA 2. *Under the hypotheses of the theorem, for each n there is a family $\rho_{n,k}$ of seminorms defining the topology T_n such that*
(1) $\rho_{n,k}(z)\leqq\rho_{n,k+1}(z)\quad \forall k, \forall n$, *and*
(2) $\rho_{n,k}(i_n^*(z))\leqq\rho_{n+1,k}(z)\quad \forall k, \forall n$.

PROOF. Choose $\rho_{1,k}$ on E_1' satisfying (1) and defining the topology T_1. Since i_1^* is continuous with respect to T_2 and T_1, if $V_m(0)$ is a decreasing base of absolutely convex neighborhoods for E_2', there is a subsequence $V_{m_k}(0)$ such that

$$\rho_{1,k}(i_1^*(z))\leqq 1\quad \text{for all } z\in V_{m_k}$$

and $V_{m_k}\subset V_{m_{k-1}}$. Let $\rho_{2,k}$ be the seminorm determined by V_{m_k}. Then

$$\rho_{2,k}(z)\leqq\rho_{2,k+1}(z),\quad\text{and}\quad \rho_{1,k}(i_1^*(z))\leqq\rho_{2,k}(z).$$

It is clear that this process may be repeated by induction so as to yield the conclusion of the lemma.

PROOF OF THEOREM 1. For each n, let $\rho_{n,k}$ be a sequence of seminorms defining the topology T_n for E_n' satisfying (1) and (2) in Lemma 2. By Lemma 1, $i_n^*(F_{n+1}^\circ)$ is $\sigma(E_n', E_n)$-dense in F_n°, hence $\tau(E_n', E_n)$-dense in F_n°, and therefore T_n-dense in F_n°. For each n, let f_n be a continuous extension of φ_n to E_n. Setting $z_1=0$ and using the T_n-denseness of $i_n^*(F_{n+1}^\circ)$ in F_n°, we may construct a sequence z_n such that

$$z_n\in F_n^\circ, \tag{1}$$

$$\rho_{n-1,n-1}(f_{n-1}+z_{n-1}-i_n^*(f_n+z_n))<(1/2)^{n-1}. \tag{2}$$

Setting

$$\delta_n = f_n + z_n - i_n^*(f_{n+1} + z_{n+1}), \tag{3}$$

let

$$g_n = f_n + z_n - \delta_n - i_n^*(\delta_{n+1}) - i_n^* \circ i_{n+1}^*(\delta_{n+2}) - \cdots .$$

The series for g_n converges in E'_n since given $\rho_{n,m}$,

$$\begin{aligned}\rho_{n,m}(i_n^* \circ \cdots \circ i_{n+r}^*(\delta_{n+r+1})) &\leqq \rho_{n+1,m}(i_{n+1}^* \circ \cdots \circ i_{n+r}^*(\delta_{n+r+1})) \\ &\leqq \cdots \leqq \rho_{n+r,m}(i_{n+r}^*(\delta_{n+r+1})) \leqq \rho_{n+r+1,m}(\delta_{n+r+1}) \\ &\leqq \rho_{n+r+1,n+r+1}(\delta_{n+r+1}) \leqq \frac{1}{2^{n+r+1}} \quad \text{if } m \leqq n+r+1.\end{aligned}$$

Hence, the sequence of partial sums is Cauchy, and since (E'_n, T_n) is complete, $g_n \in E'_n$ is defined. Define $\hat{\varphi}(x) = g_n(x)$, $x \in E_n$. Then $\hat{\varphi}$ is a well-defined linear functional on E since, by (3),

$$\begin{aligned} i_n^*(g_{n+1}) &= i_n^*(f_{n+1}) + i_n^*(z_{n+1}) - i_n^*(\delta_{n+1}) - i_n^* \circ i_{n+1}^*(\delta_{n+2}) - \cdots \\ &= f_n + z_n - \delta_n - i_n^*(\delta_{n+1}) - i_n^* \circ i_{n+1}^*(\delta_{n+2}) - \cdots \\ &= g_n. \end{aligned}$$

Moreover, $\hat{\varphi}|_{E_n} = g_n$ and hence is continuous, while $\hat{\varphi}|_{F_n} = g_n|_{F_n} = f_n|_{F_n} = \varphi_n$. Hence $\hat{\varphi}|_F = \varphi$.

REMARK 1. One may readily find examples to which Theorem 1 applies through the following observation. Let E_n be an increasing sequence of locally convex spaces such that for each n, there is a Fréchet space G_n with topology T_n and $G_n^{\#} \supseteq E_n \supseteq G'_n$, where $^{\#}$ and $'$ indicate the algebraic and continuous duals respectively. If E_n is endowed with any topology S_n of the dual pair (E_n, G_n), then $E'_n = G_n$ and $\tau(E'_n, E_n) \geqq T_n$. Let the canonical injections $i_n: E_n \to E_{n+1}$ be S_n, S_{n+1} continuous and suppose $i_n(G'_n) \subset G'_{n+1}$ (this last condition is automatically satisfied if $E_n = G'_n$). Since $\sigma(E_n, E'_n)|_{G'_n} = \sigma(G'_n, G_n)$, $i_n|_{G'_n}$ is continuous with respect to the topologies $\sigma(G'_n, G_n)$ and $\sigma(G'_{n+1}, G_{n+1})$. Thus $i_n^*: G_{n+1} \to G_n$ is continuous with respect to $\tau(G_{n+1}, G'_{n+1})$ and $\tau(G_n, G'_n)$, that is, with respect to T_{n+1} and T_n. We see then that (2) and (3) of Theorem 1 are both satisfied in this context.

We now give a specific illustration of the situation described in Remark 1.

PROPOSITION 1. *Let F_n be a sequence of Fréchet spaces, and for each n let F'_n have a locally convex topology compatible with the pairing (F'_n, F_n). Let $E_n = \prod_{k=1}^n F'_n$ be endowed with the product topology S_n and let $E = \bigcup_{n=1}^\infty E_n$ be the inductive limit of the sequence E_n (i.e. E is the locally* *convex sum of the F'_n). Suppose M is a subspace of E such that for all n, $M_n = M \cap E_n$ is closed in E_n. If φ is a linear functional on M such that for all n, $\varphi|_{M_n}$ is continuous, then φ has a continuous linear extension to E.*

PROOF. Set $G_n = \prod_{k=1}^n F_k$ and apply the reasoning of Remark 1.

REMARK 2. The theorem yields no new information in the case that the E_n are Banach spaces. This follows by the fact that in E'_n any Fréchet topology T_n weaker than $\tau(E'_n, E_n)$ is weaker than $\beta(E'_n, E_n)$, the (complete) norm topology. Hence, by the Baire Category Theorem $T_n = \beta(E'_n, E_n) = \tau(E'_n, E_n)$, and so E_n is reflexive.

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