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ON THE DOMINATED ERGODIC THEOREM IN L_2 SPACE

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ABSTRACT. Let T be a contraction on L_2 of a σ-finite measure space, $A_n(T)$ the operator $(1/n)(T^0+\cdots+T^n)$, $S(T)f$ the function $\sup_n|A_n(T)f|$. THEOREM 1. Assume that, whatever be the measure space, $S(U)f\in L_2$ for each unitary operator U on L_2 and each function $f\in L_2$. Then there exists a universal constant K such that $\|S(T)f\|\leqq K\|f\|$ for each contraction T on L_2 and each $f\in L_2$. THEOREM 2. Let T be a contraction on L_2 and let U be a unitary dilation of T acting on a Hilbert space H containing L_2. If all expressions of the form $\sum_{n=1}^{\infty} P_nA_n(U)$, where P_n are mutually orthogonal projections, are bounded operators on H, then for each $f\in L_2$, $S(T)f\in L_2$ and $A_n(T)f$ converges a.e.

Let T be a contraction on L_2 of a σ-finite measure space $(X, \mathscr{F}, \mu)$. $A_n(T)$ is the operator $(1/n)(T^0+\cdots+T^n)$. If f is a function in L_2, let $S(T)f$ be the function $\sup_{n>0}|A_n(T)f|$. One of the unresolved problems of ergodic theory is whether $A_n(T)f$ converges almost everywhere for each f in L_2. It is known that the result would be implied by the *dominated ergodic theorem*; i.e., the existence of a constant K such that for all $f\in L_2$

$$\|S(T)f\| \leqq K \|f\|. \tag{1}$$

We show below that if the dominated ergodic theorem holds for unitary operators ($\equiv$invertible isometries) on L_2, then it holds for contractions. Whether the theorem holds for unitary operators on L_2 is still not known. It may be pointed out that the dominated ergodic theorem holds for *positive* unitary operators (but perhaps not for positive contractions) on L_2, as shown by E. M. Stein (see [**4**, p. 367] and [**5**, p. 87]), and for invertible, not necessarily positive, isometries on L_p, $1<p<\infty$, $p\neq 2$, as shown by Mrs. A. Ionescu Tulcea [**4**]. (Positive means $f\geqq 0$ implies

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$Tf \geqq 0$ a.e.) For positive contractions on L_p, partial results were obtained by R. V. Chacon and J. Olsen [**2**], and Chacon and S. M. McGrath [**1**].

We also prove a result about *individual* contractions which may be of interest should the general conjecture prove to be false. Given an operator U, call V a *decomposition* of U iff V is of the form $\sum_{n=1}^{\infty} P_n A_n(U)$, where P_n are mutually orthogonal projections. Let T be a contraction on L_2 and U a unitary dilation of T acting on a space $H \supseteq L_2$. We prove that if all decompositions of U are bounded operators on H then $S(T)f \in L_2$ for each $f \in L_2$, and hence $A_n(T)f$ converges almost everywhere.

THEOREM 1. *Assume that, whatever be the space L_2, $S(U)f \in L_2$ for each unitary operator U on L_2 and each function $f \in L_2$. Then there exists a constant K such that* (1) *holds for each contraction T on L_2 and each $f \in L_2$.*

PROOF. We first observe that there is a constant K such that

$$\|S(U)f\| \leqq K \|f\| \tag{2}$$

for all $f \in L_2$ and all unitary operators U on L_2. Otherwise for $n=1, 2, \cdots$, there would exist measure spaces $(X_n, \mathscr{F}_n, \mu_n)$, unitary operators U_n on $L_2(X_n, \mathscr{F}_n, \mu_n)$, and functions $g_n \in L_2(X_n, \mathscr{F}_n, \mu_n)$, such that $\|g_n\| = 1/n$ and $\|S(U_n)g_n\| \geqq 1$. Let $(X, \mathscr{F}, \mu)$ be the direct sum $\bigoplus_{n=1} (X_n, \mathscr{F}_n, \mu_n)$, and represent a function f on X as $f=(f_n)$, where f_n is the restriction of f to X_n. Define a unitary operator U on $L_2(X, \mathscr{F}, \mu)$ by $U(f_n)=(U_n f_n)$. Then

$$g = (g_n) \in L_2(X, \mathscr{F}, \mu),$$

but

$$\|S(U)g\|^2 = \left\| \sum_{n=1}^{\infty} S(U_n) g_n \right\|^2 = \infty,$$

which is a contradiction. Thus (2) holds. Now let $E_1, E_2, \cdots$, be disjoint measurable sets; write 1_E for the indicator function of a set E. For each unitary operator U, each $f \in L_2$

$$\left| \sum_{n=1}^{\infty} 1_{E_n} A_n(U) f \right| \leqq S(U) f \quad \text{a.e.,} \tag{3}$$

which implies

$$\left\| \sum_{n=1}^{\infty} 1_{E_n} A_n(U) \right\| \leqq K, \tag{4}$$

where in (4) 1_{E_n} are projection operators corresponding to the multiplication by 1_{E_n}, and K is the constant appearing in (2). Next observe that (4) may be generalized to

$$\left\| \sum_{n=1}^{\infty} P_n A_n(U) \right\| \leqq K, \tag{5}$$

where U is a unitary operator on an arbitrary Hilbert space H, (P_n) is any sequence of mutually orthogonal projections on H, and K is again the constant in (2). This follows from the fact that given H and (P_n), there exists $L_2(X, \mathscr{F}, \mu)$ isometrically isomorphic to H and such that under the isomorphism, P_n becomes 1_{E_n}. Now let T be any contraction on L_2 of an arbitrary measure space $(X, \mathscr{F}, \mu)$. Then

$$\left\| \sum_{n=1}^{\infty} 1_{E_n} A_n(T) \right\| \leqq K \tag{6}$$

for any sequence of disjoint sets E_n in $\mathscr{F}$, where K is as before. To prove (6), apply the *dilation theorem* of Sz.-Nagy (cf. [**6**] or [**3**]; *dilations* in [**6**] are *power dilations* in the terminology of [**3**]): there exists a Hilbert space H, a projection P, and a unitary operator U on H such that $PH = L_2(X, \mathscr{F}, \mu)$ and $T^n = PU^n$ for $n=1, 2, \cdots$. Then $A_n(T) = PA_n(U)$, hence $1_{E_n}A_n(T) = 1_{E_n}PA_n(U)$. $P_n = 1_{E_n}P$ form a set of mutually orthogonal projections, and therefore (6) follows from (5). Finally, to conclude the proof of the theorem, note that given T and f there exist mutually disjoint sets $E_1, E_2, \cdots$ such that $S(T)f = |\sum_{n=1}^{\infty} 1_{E_n} A_n(T)f|$. Hence

$$\|S(T)f\| \leqq \left\| \sum_{n=1}^{\infty} 1_{E_n} A_n(T) \right\| \cdot \|P\| \cdot \|f\| \leqq K\|f\|. \quad \square \tag{7}$$

Now consider the case of an individual contraction T.

THEOREM 2. *Let T be a contraction on $L_2(X, \mathscr{F}, \mu)$ and let U be a unitary dilation of T acting on a Hilbert space H. If all decompositions of U are bounded operators on H, then, for each $f \in L_2$, $S(T)f \in L_2$ and $A_n(T)f$ converges a.e.*

PROOF. Let $f \in L_2$, $E_n = \{x \in X : |A_n(T)f| = S(T)f\}$, $P_n = 1_{E_n}P$, where P is a projection such that $T^n = PU^n$, $n = 1, 2, \cdots$. Disjoint the sets E_n if necessary. If there exists a constant $C = C(T, f)$ such that

$$\left\| \sum_{n=1}^{\infty} P_n A_n(U) \right\| \leqq C,$$

then the relation (7) holds with C replacing K, showing that $S(T)f \in L_2$. The proof of the pointwise convergence of $A_n(f)$ is like in [**1**] or [**4**]. $\square$

REMARK. It is clearly of interest to consider only the *minimal* unitary dilation U_0 of T, since each unitary dilation reduces to U_0 on $\bigvee_{n=-\infty}^{+\infty} U_0^n L_2$ (cf. [**6**]).

We finally note that sequences of successive Cesàro averages of powers of operators in Theorems 1 and 2 may be replaced by sequences of any linear combinations of powers; the proofs remain the same.

ADDED IN PROOF. Professor D. L. Burkholder has pointed out to us that a result of his, a consequence of his theory of semi-Gaussian spaces (Theorem 2, p. 128, Trans. Amer. Math. Soc. **104** (1962), implies that the dominated ergodic theorem fails for contractions on L_2. Combining this with the results of the present paper, one obtains that the answer to the question about unitary operators raised in the introduction is negative: There exists a unitary operator on L_2 for which the dominated ergodic theorem fails.

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