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FUNCTIONS WITH A CLOSED GRAPH[1]

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ABSTRACT. Let X be a T_2 Baire space. A set $F\subset X$ is closed and nowhere dense in X if F is the set of points of discontinuity of a function with a closed graph from X into R^n. Although the converse does not hold in general, it does hold when X is the real line.

1. **Introduction.** Let X and Y be topological spaces and let f be a function from X into Y. Put $D(f)=\{x\in X|f \text{ is discontinuous at } x\}$. f has a closed graph if $\{(x,f(x))|x\in X\}$ is closed in $X\times Y$. R^n is used to denote Euclidean n-space. It is well known (see [**3**, p. 78]) that in order for $F\subset X$ to coincide with the set of points of discontinuity of a real-valued function on X, it is necessary that F be an F_σ set without isolated points. It is shown in [**1**] that this condition is also sufficient for a wide class of topological spaces.

In this note it is shown that if X is a Baire space which is also Hausdorff and if f is a function from X into R^n with a closed graph, then $D(f)$ is a closed and nowhere dense subset of X (Theorem 2). It is also shown that a set $F\subset R$ is closed and nowhere dense in R if and only if there exists a function $f: R\to R$ with a closed graph such that $D(f)=F$ (Theorem 3). This theorem cannot be extended to arbitrary T_2 Baire spaces (Example 2).

2. **The main results.** The following theorem is known (see for example [**2**, p. 228]).

THEOREM 1. *Let X be a Hausdorff space and let Y be compact. Then $f: X\to Y$ is continuous if and only if the graph of f is closed.*

The following two lemmas will be useful in establishing the main results.

LEMMA 1. *Let X be a Hausdorff space and let Y be a metric space in which each bounded set has a compact closure. If $f: X\to Y$ is a function with a closed graph, then $D(f)$ is a closed subset of X.*

PROOF. For each $x\in X$, put

$$\omega(x) = \inf\{\operatorname{diam} f(U) \mid U \text{ is a neighbourhood of } x\}.$$

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Suppose there exists some $x \in D(f)$ such that $\omega(x)=L$, where $0<L<+\infty$. Let $\varepsilon>0$, then there exists an open neighbourhood U of x such that $L-\varepsilon<\operatorname{diam} f(U)<L+\varepsilon$. Hence, $f(U)$ is contained in a compact subset of Y and this implies, by Theorem 1, that $f|U$ is continuous on U. However, U is a neighbourhood of x, therefore, f is continuous at x. But this is impossible, since $x \in D(f)$. Therefore, for each $x \in D(f)$, $\omega(x)=+\infty$. (If $x \notin D(f)$ it is easily seen that $\omega(x)=0$.) Let $\alpha \in R$. Then $\{x \in X \mid \omega(x)<\alpha\}$ is an open subset of X (see [**3**, p. 78]). Hence $D(f)=\{x \in X \mid \omega(x)=+\infty\}$ is a closed subset of X.

REMARK 1. Let X be a Hausdorff space and let $f: X \rightarrow R^n$ be a function with a closed graph. Let $x \in D(f)$. It follows from the proof of the preceding lemma that $\omega(x)=+\infty$. Therefore, f is unbounded on every neighbourhood of x.

LEMMA 2. *Let X be a Baire space which is Hausdorff. If $f: X \rightarrow R^n$ is a function with a closed graph, then $D(f)$ is a nowhere dense subset of X.*

PROOF. Suppose there exists an open set $U \subset X$ such that $U \subset D(f)$. It follows from Lemma 1 that $\bar{U} \subset D(f)$. $\bar{U}$ is of second category since, in a Baire space, a set of first category has no interior (see [**2**, p. 250]). For each positive integer m, let $B_m=\{x \in \bar{U} \mid |f(x)| \leqq m\}$, where $|\cdot|$ denotes the usual Euclidean norm on R^n.

For each integer m, B_m is closed. Suppose this is not true. Then for some m there exists $x \in \bar{B}_m$ such that $x \notin B_m$. Let $\{x_\alpha\}_{\alpha \in A}$ be a net converging to x such that $x_\alpha \in B_m$ for all $\alpha \in A$. Put $N=\{x\} \cup \{x_\alpha \mid \alpha \in A\}$. Then $f(N) \subset K$, where K is a compact subset of R^n. Since N is a closed subset of X and since X is Hausdorff, $f|N$ has a closed graph in $N \times K$ (and also in $X \times R^n$). Therefore, $f|N$ is continuous on N, by Theorem 1, and $f|N(x_\alpha) \rightarrow f|N(x)$. This implies that $f(x_\alpha) \rightarrow f(x)$. Since $|f(x_\alpha)| \leqq m$, for all $\alpha \in A$, it follows that $|f(x)| \leqq m$. This contradicts the assumption that $x \notin B_m$. Hence, for each integer m, B_m is a closed subset of X.

Since $\bigcup_{m=1}^{\infty} B_m=\bar{U}$ and since $\bar{U}$ is of second category in X, it follows that, for some integer m_1, there exists an open set V (open in X) such that $V \subset \bar{V} \subset B_{m_1}$. Again, $f|\bar{V}$ is a bounded function on $\bar{V}$ and hence $f|\bar{V}$ is continuous on $\bar{V}$. This implies that f is continuous at each point of V. This contradicts the assumption that $V \subset \bar{U} \subset D(f)$. Therefore, $D(f)$ is a nowhere dense subset of X.

The following theorem is an immediate consequence of the preceding two lemmas.

THEOREM 2. *Let X be a Baire space which is Hausdorff. If $f: X \rightarrow R^n$ has a closed graph, then $D(f)$ is a closed and nowhere dense subset of X.*

In general, closed and nowhere dense subsets of a T_2 Baire space cannot be characterized as the points of discontinuity of a real-valued function with a closed graph (see Example 2). The next theorem shows this characterization does hold in a special case.

THEOREM 3. *A set $F \subset R$ is closed and nowhere dense if and only if there exists a function $f: R \to R$ such that f has a closed graph and $D(f)=F$.*

PROOF. The sufficiency of the condition follows from Theorem 2. Conversely, if $F=\varnothing$, the theorem is immediate. So, we may assume that $F \neq \varnothing$. $F^c = \bigcup_{n=1}^{\infty} I_n$, where $I_n \cap I_m = \varnothing$, if $n \neq m$, and $I_n=(a_n, b_n)$, for $n=1, 2, \cdots$. For each n, let m_n be the midpoint of the open interval I_n. Define a function $f: R \to R$ as follows:

$$\begin{aligned} f(x) &= n(m_n - a_n)/(x - a_n), && \text{if } x \in (a_n, m_n], \text{ for } n = 1, 2, \cdots, \\ &= 0, && \text{if } x \in F, \\ &= n(b_n - m_n)/(b_n - x), && \text{if } x \in [m_n, b_n), \text{ for } n = 1, 2, \cdots. \end{aligned}$$

Then f is well defined, f is continuous at each point of F^c, and f is discontinuous at each point of F, since F is closed and nowhere dense in R.

It remains to be shown that the graph of f is closed. If $x \in F^c$, then, since f is continuous on F^c, (x, y) is a limit point of the graph of f only if $y = f(x)$. Let $p \in F$ and $0 \neq y \in R$. Let k be a positive integer such that $-k<y<k$. If $x \in \bigcup_{n>k}^{\infty} I_n$, then $|f(x)|>k$. Therefore, there exists a neighbourhood N_1 of (p, y) such that $(x, f(x)) \notin N_1$, for all $x \in \bigcup_{n>k}^{\infty} I_n$. It follows from the construction of f that there exists a neighbourhood N_2 of (p, y) such that $(x, f(x)) \notin N_2$, for $x \in \{\bigcup_{n=1}^{k} I_n\} \cup F$. Hence the graph of f is closed in $R \times R$.

REMARK 2. If F is a nowhere dense perfect subset of R, then the function f constructed in the preceding theorem has a closed graph and has a discontinuity of the second kind at each point of F. [That is, if $a \in F$, then either $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ does not exist.]

3. We now give three examples to indicate some of the restrictions encountered in an attempt to extend Theorem 2.

EXAMPLE 1. In Theorem 2 we cannot omit the condition that X is a Baire space. Let X be the space of rational numbers with the topology inherited from R. Let $\{\gamma_n | n=1, 2, \cdots\}$ be an enumeration of X. Define $f: X \to R$ by $f(\gamma_n)=n$. Then the graph of f is closed in $X \times R$ and $D(f)=X$.

EXAMPLE 2. There exists a compact Hausdorff space X (hence, a Baire space) and a closed nowhere dense subset F of X such that, if $f: X \to R$ is a function with a closed graph, then $D(f) \neq F$. Let X be the space of all ordinals less than or equal to the first uncountable ordinal, Ω, with the order topology. Put $F=\{\Omega\}$. Let $f: X \to R$ be any function with a closed graph. $X-F$ is countably compact, so, if f is continuous at each point of $X-F$, f must be bounded on $X-F$. This implies, by Remark 1, that $D(f) \neq F$.

EXAMPLE 3. In Theorem 2, we cannot replace R^n with an arbitrary metric space. Let X denote the real line with the usual metric and let Y denote the real line with the discrete metric. Let f be the identity function from X into Y. Then f has a closed graph and $D(f)=X$.

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