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CHARACTERIZATION OF FINITE-DIMENSIONAL Z-SETS

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ABSTRACT. It is proved that closed finite-dimensional subsets of Q and l_2 are Z-sets iff their complement is 1-ULC. As a corollary, closed finite-dimensional sets of deficiency 1 are shown to be Z-sets.

0. **Introduction.** J. L. Bryant and C. L. Seebeck have proved a homeomorphism extension theorem for k-dimensional compacta in R^n with 1-ULC complements, where $2k+2\leqq n$ (see [3], [4]). Their results have been considerably generalized by M. A. Štan'ko. Štan'ko gives in [8] several definitions of "dimension-of-embedding" for closed subsets of R^n and proves, besides equivalence of these definitions, the following result:

THEOREM (ŠTAN'KO). *If K is a closed subset of R^n and* $\dim(K)=k\leqq n-3$, *then the dimension-of-embedding of K equals k iff $R^n\backslash K$ is* 1-*ULC. Otherwise it is equal to $n-2$. If* $\dim(K)\geqq n-2$, *then the dimension-of-embedding coincides with ordinary dimension.*

If the dimension gap between K and R^n is sufficiently large, then equality of both dimensions can be considered as a definition of tame embeddings. This apparatus cannot distinguish between tame and wild arcs in R^3, because the dimension gap is too small.

Professor R. D. Anderson suggested to me that some generalization to the infinite-dimensional case might be possible. An intuitive rephrasing of Štan'ko's result is: If $R^n\backslash K$ is 1-ULC then $R^n\backslash K$ is locally and globally homotopically trivial up to as high a dimension as is compatible with the dimension of K. Stated this way, the obvious generalization to the cases $X=l_2$ and $X=Q$ becomes: if K is a finite-dimensional closed subset of X, then K is a Z-set in X iff $X\backslash K$ is 1-ULC. This is the main theorem of this paper. The proof is a straightforward generalization of Štan'ko's proof of Proposition 5 in [8], applied to the infinite-dimensional case. However, no knowledge of infinite-dimensional topology is needed to follow the argument.

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1. **Definitions.** A closed subset K of a space X is a *Z-set* iff for every nonempty homotopically trivial open subset O of X, $O\backslash K$ is nonempty and homotopically trivial.[2] A map $f:X\to Y$ with Y metric is called *ε-small* if the diameter of $f(X)$ is at most ε. A metric space Y is k-ULC (*k-uniformly locally connected*) if for all ε there exists a δ such that every δ-small map $f:S^k\to Y$ can be extended to an ε-small map $\bar{f}:I^{k+1}\to Y$, where S^k is the combinatorial boundary of I^{k+1}. If we define $S^{-1}=\varnothing$ and $I^0=\{0\}$ then (-1)-ULC means nonempty.

In special cases an alternative definition is possible. For this definition we use the term k-ULC$^-$ instead of k-ULC. We shall work with $s=(-1,1)^\infty$ rather than with l_2. In **[1]** it is proved that $s\simeq l_2$. For $X=Q=[-1,1]^\infty$ or $X=s$, an *open cube* in X is a basis element of the product topology, i.e., a product of relatively open subintervals of $[-1,1]$ or $(-1,1)$ resp., such that only finitely many (maybe none) are different from the whole interval. In analogy to **[8]**, we define: if K is a closed subset of X then $X\backslash K$ is k-ULC$^-$ iff for every open cube $A\subset X$ every map $f:S^k\to A\backslash K$ can be extended to a map $\bar{f}:I^{k+1}\to A\backslash K$. This definition is independent of the metric of X, but it refers instead to the embedding of K into X. For $X=Q$ or $X=s$, if K is k-ULC$^-$ then K is k-ULC (it is only necessary to find a reasonably small open cube containing $f(S^k)$ for any given $f:S^k\to X$) but the converse does not generally hold. However, we can prove the following:

Lemma 1.1. *If $X=l_2$ or $X=Q$ and K is a closed finite-dimensional subset of X then $X\backslash K$ is 1-ULC$^-$ iff $X\backslash K$ is 1-ULC.*

Proof. As remarked above, the former implies the latter. The proof of the converse is straightforward but tedious. Let $A\subset X$ be an open cube and let $f:S^1\to A\backslash K$ be given. Let $F:I^2\to A$ be any extension of f. Let $\varepsilon=\frac{1}{2}d(F(I^2), X\backslash A)$ (not the Hausdorff distance). Choose $\delta<\varepsilon$ such that every δ-small map $h:S^1\to X\backslash K$ can be extended to an ε-small map $\bar{h}:I^2\to X\backslash K$. Now every less than $\delta/2$-small $g:S^0\to X\backslash K$ can be extended to a $\delta/2$-small map $\bar{g}:I\to X\backslash K$. Choose $\xi<\delta/6$ such that for every $x, y\in I^2$, $d(x,y)<\xi$ implies $d(F(x),F(y))<\delta/6$. Let T be a ξ-fine simplicial subdivision of I^2 with i-skeletons T_i, $i=0,1,2$. Choose $F_0:T_0\to X\backslash K$ with

$$\max_{x\in T_0} d(F_0(x), F(x)) < \delta/6 \quad\text{and}\quad F_0|_{T_0\cap S^1} = f|_{T_0\cap S^1}.$$

Then, for adjacent $x, x'\in T_0$, $d(F_0(x), F_0(x'))<\delta/6+2\cdot\delta/6=\delta/2$. Moreover $F_0(T_0)$ is contained in a $\delta/6$-neighborhood of $F(I^2)$. So we can connect

[2] For ANR's and in particular for open subsets of Q, homotopic triviality or contractibility is equivalent to triviality of all homotopy groups in positive dimensions (Palais **[6]**).

$F_0(x)$ and $F_0(x')$ by a $\delta/2$-small arc in $X\backslash K$. Thus we find $F_1: T_1 \to X\backslash K$ with $F_1(T_1)$ contained in a $(\delta/2+\delta/6)$-neighborhood of $F(I^2)$ and such that $F_1|_{S^1}=f$. For each 2-simplex $\Delta^{(j)}$ in T_2, $F_1|_{\partial\Delta^{(j)}}$ is δ-small, hence can be extended to an ε-small map $F_2^{(j)}:\Delta^{(j)}\to X\backslash K$. Now $F_2=\bigcup_j F_2^{(j)}:I^2\to X\backslash K$ is the required extension of f. The only thing left to be proved is that $F_2(I^2)\subset A$. But $F_2(I^2)$ is contained in an $(\varepsilon+\delta/2+\delta/6)$-neighborhood of $F(I^2)$ and since $\varepsilon+\delta/2+\delta/6<\varepsilon+\delta<2\varepsilon$, it follows by choice of ε that $F_2(I^2)$ is contained in A.

2. Theorems.

LEMMA 2.1. *For $X\cong Q$ or $X\cong s$, if $A\subset X$ is an open cube and K is a finite-dimensional closed subset of X, then $A\backslash K$ has the homology of a point.*

PROOF. As we shall see, this is a consequence of the Alexander duality theorem (see e.g. [7])[3] and the fact that $H^q(K)=0$ if $q>k=\dim(K)$. Consider the set $s_f=\{x\in s\subset Q \mid x_i=0$ for all but finitely many $i\}$. This can be written as $\bigcup_n s_n$, with $s_n\cong R^n$, e.g., $s_n=\{x\mid x_i=0$ if $i>n$ and $|x_i|<1-1/n$ for $i=1,\cdots,n\}$. Define

$$g_{n,t}(x)=(1-t)\cdot x+t\cdot(1-1/2n)\cdot(x_1,\cdots,x_n,0,0,\cdots).$$

Then every map $\varphi:T\to s$ or $\varphi:T\to Q$, where T is any topological space, is homotopic to a map $\varphi'=g_{n,1}\circ\varphi:T\to s_n$ by a homotopy $(g_{n,t}\circ\varphi)_t$. The sets $\{g_{n,t}\circ\varphi(x):t\in[0,1]\}$ can be made uniformly small by choosing n sufficiently large. Let $\bigcup_n C_n$ be a corresponding set in $A\subset X$ and $\{(h_{n,t})_t\}_n$ be a corresponding family of homotopies such that $(h_{n,t})_t$ contracts A into C_n.

Let $T=\sum_i \lambda_i T_i$ be a q-cycle in $A\backslash K$. We show that it bounds in $A\backslash K$. If $\dim(K)=k$, then $H^m(K)=\{0\}$ for $m>k$. Because $K\cap C_n$ is a closed subset of C_n of dimension $\leq k$, by Alexander Duality we infer that for $n>k+q+1$, $H_q(C_n\backslash K)$ is trivial. For sufficiently large $m>k+q+1$, the cycle $T'=\sum_i \lambda_i h_{n,1}\circ T_i$ is a cycle in $C_m\backslash K$. Then T' also bounds in $A\backslash K$, and, using the homotopy $(h_{n,t})_t$, it is easily seen that T bounds in $A\backslash K$: specifically, define $T_i^I:\Delta_q\times I\to A\backslash K$ by $T_i^I(p,t)=h_{n,t}\circ T_i(p)$; let $S=\sum_j S_j$, with $S_j:\Delta_{q+1}\to\Delta_q\times I$ be a triangulation of $\Delta_i\times I$ such that ∂S includes among its terms $\tilde{S}_0-\tilde{S}_1$, where $\tilde{S}_i$ is the obvious map from Δ_q onto $\Delta_q\times\{i\}$. Let

[3] One form of the Alexander duality theorem states that $\tilde{H}_q(R^n\backslash A)\cong H^{n-q-1}(A)$ for A compact and $\tilde{H}_q$ denoting reduced singular homology. In case A is only closed and not compact, one can form the one-point compactification of R^n and A and remove a point $p\notin A$ from R^n. Then $A\cup\{\infty\}$ is a compact subset of $(R^n\cup\{\infty\})\backslash\{p\}\cong R^n$, and $A\cup\{\infty\}$ has the same dimension as A, and except maybe in the dimensions $n-1$ and n, $[(R^n\cup\{\infty\})\backslash\{p\}]\backslash(A\cup\{\infty\})=(R^n\backslash\{p\})\backslash A$ has the same homology groups as $R^n\backslash A$. Hence for a noncompact closed subset A of R^n and for $q<n-1$ we have $\tilde{H}_q(R^n\backslash A)\cong H^{n-q-1}(A\cup\{\infty\})$.

$T' = \partial\tilde{T}$; then

$$\partial\Big(\sum_{i,j}\lambda_i \cdot T_i^I \circ S_j\Big) = T - T' \quad \text{and} \quad T = \partial\tilde{T} + \partial\Big(\sum_{i,j}\lambda_i T_i^I \cdot S_j\Big).$$

The following lemma is well known in the folklore. We will give a formal proof.

LEMMA 2.2. (a) *If $\mathscr{B}$ is a base for Q consisting of homotopically trivial open sets, then a closed subset K of Q is a Z-set in Q iff for every $B \in \mathscr{B}$, $B\backslash K$ is nonempty and homotopically trivial.*

(b) *If $\mathscr{B}$ is the base for s consisting of all open cubes, then a closed subset K of s is a Z-set in s iff for every $B \in \mathscr{B}$, $B\backslash K$ is homotopically trivial.*

PROOF OF (a). Suppose $\mathscr{B}$ and K satisfy the conditions. Let O be a homotopically trivial open subset of Q and let $f: S^{q-1} \to O\backslash K$ be a map. We want an extension $\bar{f}: I^q \to O\backslash K$ whereas we have an extension $g: I^q \to O$ (due to homotopic triviality of O).

Cover $g(I^q)$ by a finite cover $\mathscr{B}_1 \subset \mathscr{B}$ such that for each $B \in \mathscr{B}_1$, $B \subset O$. There exists a closed neighborhood V_1 of $g(I^q)$ which is also covered by $\mathscr{B}_1$. Let ε_1 be a Lesbesgue-number for $\mathscr{B}_1$ as a covering of V_1 (i.e., each subset of V_1 with diameter less than ε_1 is contained in some element of $\mathscr{B}_1$). Define the *mesh* $m(\mathscr{A})$ of a collection $\mathscr{A}$ as the supremum of the diameters of the elements of $\mathscr{A}$. Let $\mathscr{B}_2 \subset \mathscr{B}$ be a finite covering of $g(I^q)$ with $\bigcup \mathscr{B}_2 \subset V_1$ and with $m(\mathscr{B}_2) < \varepsilon_1/2$. There exists a closed neighborhood V_2 of $g(I^q)$ which is also covered by $\mathscr{B}_2$. Again let ε_2 be a Lebesgue-number for $\mathscr{B}_2$ as a covering of V_2. In this way, construct inductively a sequence of finite coverings $\mathscr{B}_1 \subset \mathscr{B}$, $\mathscr{B}_2 \subset \mathscr{B}, \cdots, \mathscr{B}_q \subset \mathscr{B}$ with Lebesgue numbers $\varepsilon_1, \cdots, \varepsilon_q$ with respect to closed neighborhoods $V_1, \cdots, V_q$ of $g(I^q)$ and such that $\bigcup \mathscr{B}_{i+1} \subset V_i$ and $m(\mathscr{B}_{i+1}) < \varepsilon_i/2$. Because g is uniformly continuous there exists a $\delta > 0$ such that for $x, x' \in I^q$ and $d(x, x') < \delta$, $d(g(x), g(x')) < \varepsilon_q/3$. Let P be a simplicial subdivision of I^q of mesh smaller than δ. Let P_i be the i-skeleton of P, $i = 0, 1, \cdots, q$. Because for every $B \in \mathscr{B}$, $B\backslash K$ is nonempty, it follows that K is nowhere dense. Then there exists a mapping $\bar{f}_0: P_0 \to \bigcup \mathscr{B}_q\backslash K$ with $d(g|_{P_0}, \bar{f}_0) < \varepsilon_q/3$ and $\bar{f}_0|_{P_0 \cap S^{q-1}} = f|_{P_0 \cap S^{q-1}}$. Now for adjacent vertices $p, r \in P_0$,

$$d(\bar{f}_0(p), \bar{f}_0(r)) \leqq d(\bar{f}_0(p), g(p)) + d(g(p), g(r)) + d(\bar{f}_0(r), g(r)) < \varepsilon_q.$$

Because ε_q is a Lebesgue-number for $\mathscr{B}_q$, $\bar{f}_0$ maps adjacent vertices into a common element of $\mathscr{B}_q$. Now $\{B\backslash K \mid B \in \mathscr{B}_q\}$ consists of homotopically trivial sets; therefore we have an extension $\bar{f}_1: P_1 \to O\backslash K$ of $\bar{f}_0$, such that all 1-simplices are mapped into an element of $\{B\backslash K \mid B \in \mathscr{B}_q\}$, and such that $\bar{f}_1|_{P_1 \cap S^{q-1}} = f|_{P_1 \cap S^{q-1}}$. Furthermore it is easily seen that, if a mapping φ maps each face of an i-simplex onto a set of diameter $< \eta$, then φ maps the boundary of the i-simplex onto a set of diameter $<2\eta$. Observing that $m(\{B\backslash K | B \in \mathscr{B}_q\}) < \frac{1}{2}\varepsilon_{q-1}$, one sees that the boundary of every 2-simplex of P_2 is mapped onto a set of diameter $<\varepsilon_{q-1}$, which is a Lebesgue-number of $\mathscr{B}_{q-1}$ with respect to V_{q-1}. Since $\bar{f}_1(P_1) \subset \bigcup \mathscr{B}_q \subset V_{q-1}$, it follows that the image of the boundary of a 2-simplex of P_2 is contained in an element of $\mathscr{B}_{q-1}$. Using homotopic triviality of the sets $B\backslash K$ and $B \in \mathscr{B}_{q-1}$, one finds an extension $\bar{f}_2 : P_2 \to O\backslash K$ of $\bar{f}_1$ such that $\bar{f}_2|_{P_2 \cap S^{q-1}} = f|_{P_2 \cap S^{q-1}}$ and such that every 2-simplex of P_2 is mapped into a set $B\backslash K$, $B \in \mathscr{B}_{q-1}$. Repeating this procedure, we find eventually the desired extension $\bar{f} = \bar{f}_q$ of f.

PROOF OF (b). Suppose $\mathscr{B}$ and K satisfy the conditions. For B an open subset of s, define $B^* = Q\backslash(s\backslash B)^-$. Thus B^* is the largest open subset of Q such that $B^* \cap s = B$. Since $\mathscr{B}$ is the collection of *all* open cubes in s, $\mathscr{B}^* = \{B^* | B \in \mathscr{B}\}$ is a basis consisting of homotopically trivial open sets for Q. It is easily seen that $B^*\backslash \bar{K}$ is homotopically trivial if $B\backslash K$ is. Thus $\bar{K}$ is a Z-set in Q, and according to **[2]**, K is a Z-set in s.

MAIN THEOREM 2.3. *If $X \cong s$ or $X \cong Q$ and K is a finite-dimensional closed subset of X, then K is a Z-set in X iff $X\backslash K$ is* 1-*ULC.*

PROOF. Obviously the former implies the latter.

Let $X\backslash K$ be 1-ULC. According to Lemma 1.1, $X\backslash K$ is also 1-ULC$^-$. So for each open cube $A \subset X$, $A\backslash K$ has the homology of a point and has a trivial fundamental group. Now the Hurewicz theorem (see e.g. **[7]**) says that for a simply connected space Y the first nonvanishing homotopy group after π_0 is isomorphic to the first nonvanishing homology group after H_0. Applied to $A\backslash K$, this shows homotopic triviality of $A\backslash K$. By Lemma 2.3, K is a Z-set in X.

Using the standard Klee homeomorphism extension process, one can easily see that any arc in Q with deficiency 1 has property Z. In **[5]** D. W. Curtis showed moreover, using Klee-like techniques that closed finite-dimensional subsets of Q or s of sufficient deficiency have property Z. Indeed, as observed below, only deficiency 1 is needed; a result not obtainable directly using the Klee-Curtis methods.

COROLLARY 2.4. *If $X \cong s$ or $X \cong Q$ and K is a closed finite-dimensional subset of X of deficiency* 1 *(i.e., K projects onto a point in at least* 1 *coordinate), then K is a Z-set in X.*

PROOF.[4] We must show that $X\backslash K$ is 1-ULC. We may suppose that K projects onto the point 0 in the first coordinate. Let $f: S^1 \to X\backslash K$ be given.

[4] The same proof can also be used to prove the well-known theorem that if $K \subset E^n$, $\dim(K) \leq n-3$ and K has deficiency 1, then $E^n\backslash K$ is 1-ULC.

We shall find an extension which is not more than twice as large. If $f(S^1)$ meets at most one of the sets $\{x_1>0\}$ and $\{x_1<0\}$ then the existence of an extension $\bar{f}:I^2\to X\backslash K$ is trivial. So suppose $f(S^1)$ meets both sides of the hyperplane $\{x_1=0\}$.

Let π_1 denote the projection onto the first coordinate and $\bar{\pi}$ the projection onto the hyperplane $\{x_1=0\}$. There exists a map $f':S^1\to X\backslash K$ such that (1) $f'(S^1)$ is a 1-1 polygonal image of S^1, (2) for any two different vertices p and q of $f'(S^1)$, $\pi_1(p)\neq\pi_1(q)$ and $\bar{\pi}(p)\neq\bar{\pi}(q)$, and (3) f' is arbitrarily close to f. If f' is sufficiently close to f then f and f' are homotopic in $X\backslash K$ by linear interpolation. Now since all vertices of $f'(S^1)$ have a different first coordinate, only finitely many points are mapped into the hyperplane $\{x_1=0\}$. Following S^1 in either direction, number these points $p_1,\cdots,p_n$ in the order in which they are encountered. Denote the in-between intervals on S^1 by

$$T_1=[f'^{-1}(p_1),f'^{-1}(p_2)],\cdots,T_n=[f'^{-1}(p_n),f'^{-1}(p_1)].$$

Using the 0-ULC property of $\{x_1=0\}\backslash K$ (this is due to finite-dimensionality of K) and the fact that all points p_i lie outside K (since they lie on $f'(S^1)$), it can be shown that every projected arc $\bar{\pi}\circ f'(T_i)$ can be approximated by an arc $h_i(T_i)$ such that $d(\bar{\pi}\circ f'|_{T_i},h_i)$ is arbitrarily small, $h_i(T_i)\subset\{x_1=0\}\backslash K$ and $h_i(T_i)$ has endpoints p_i and p_{i+1} (or, if $i=n$, endpoints p_n and p_1). Let $h:S^1\to\{x_1=0\}\backslash K=\bigcup_i h_i$. Then h is homotopic to f' in $X\backslash K$ by linear interpolation. Next, h is homotopic to h_ε defined by $\bar{\pi}\circ h_\varepsilon=\bar{\pi}\circ h=h$ and $\pi_1\circ h_\varepsilon(x)=\varepsilon$ for all $x\in S^1$. Since $\{x_1=\varepsilon\}$ is homotopically trivial, h_ε can be extended to $\bar{h}_\varepsilon:I^2\to\{x_1=\varepsilon\}=\{x_1=\varepsilon\}\backslash K$. Hence the original map $f:S^1\to X\backslash K$ is extendable to the 2-cell, too. Moreover it will be observed that, if $f(S^1)$ has diameter $<\delta$, the extension to the 2-cell can easily be kept 2δ-small in any reasonable metric for X. This proves that $X\backslash K$ is 1-ULC, and therefore that K is a Z-set in X.

REMARK 2.5. If, for $X=Q$ or $X=s$, $X\backslash K$ is dense, 1-ULC and 0-ULC and if L is a closed subset of K, then $X\backslash L$ is 1-ULC. Thus if K is a nowhere dense closed subset of X, not necessarily a Z-set, and $X\backslash K$ is 1-ULC and 0-ULC, then every closed finite-dimensional subset of K is a Z-set in X.

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