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A PROPERTY OF TRANSFERABLE LATTICES

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ABSTRACT. A lattice K is transferable if whenever K can be embedded into the ideal lattice of a lattice L, then K can be embedded in L. An element is called doubly reducible if it is both join- and meet-reducible. In this note it is proved that every lattice can be embedded into the ideal lattice of a lattice with no doubly reducible element. It follows from this result that a transferable lattice has no doubly reducible element.

1. **Introduction.** In [5] two concepts of transferability of a lattice were introduced, named transferability and weak transferability in [2] and named sharp transferability and transferability, respectively, in this paper and [4]. A rather satisfying theory of sharp transferability can be found in [2] and [4]; see also [3] for the case of semilattices. Recently K. Baker proved that all finite projective lattices are transferable (see [1]). Still, the only known property of transferable lattices is the one announced in [5] without proof. The purpose of this note is to supply a proof of this property (see Theorem below).

First, two definitions. A lattice K is called *weakly transferable* iff whenever K can be embedded into the lattice of all ideals of a lattice L, then K can also be embedded into L. Observe, that in the papers referred to above the finiteness of K is also assumed; for the purposes of this note, however, it is not necessary to assume that K is finite. An element a of the lattice K is *doubly reducible* iff there exist elements x, y, z, u of K, all distinct from a, such that $a=x\vee y=z\wedge u$.

THEOREM. *A transferable lattice contains no doubly reducible element.*

2. **Proof of the Theorem.** Let A and B be posets. The lexicographic product of A and B, denoted by $A\otimes B$, is a poset defined on $A\times B$ with the ordering ($a, a' \in A, b, b' \in B$):

$$(1) \qquad \langle a, b\rangle \leqq \langle a', b'\rangle \quad \text{iff} \quad a < a' \quad \text{or} \quad a = a' \quad \text{and} \quad b \leqq b'.$$

In this note, let A and B be lattices and let B have a least element 0 and largest element 1.

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LEMMA 1. *$A \otimes B$ is a lattice and, for $a, a' \in A$, $b, b' \in B$, we have that*

$$(2) \qquad \langle a, b\rangle \vee \langle a', b'\rangle = \langle a \vee a', b''\rangle$$

with suitable $b'' \in B$; in fact, if a and a' are incomparable, then $b''=0$.

PROOF. Trivial; observe, that if $a<a'$, then $b''=b'$; if $a'<a$, then $b''=b$; if $a=a'$, then $b''=b\vee b'$.

Call an element p *join-reducible* if $p=x\vee y$ with $p\neq x$ and $p\neq y$; otherwise p is *join-irreducible.* The dual concepts are *meet-reducible* and *meet-irreducible.* From Lemma 1 we conclude immediately:

COROLLARY 2. *All join-reducible elements of $A\otimes B$ are of the form $\langle a, b\rangle$ where $b=0$ or b is join-reducible in B.*

Corollary 2 and its dual yield:

COROLLARY 3. *Let us assume that B has more than one element, and 0 is meet-irreducible, and 1 is join-irreducible in B. Then all doubly reducible elements of $A\otimes B$ have the form $\langle a, b\rangle$, where b is doubly reducible in B. In particular, if B has no doubly reducible element, then neither does $A\otimes B$.*

Now we map ideals of A into ideals of $A\otimes B$. Let I be an ideal of A. We set

$$(3) \qquad \bar{I} = \{\langle a, b\rangle \mid a \in I, b \in B\}.$$

LEMMA 4. *For any ideal I of A, the set $\bar{I}$ is an ideal of $A\otimes B$. The map $I\to\bar{I}$ is one-to-one, and for ideals I and J of A it satisfies*

$$(4) \qquad \bar{I}\wedge\bar{J} = (I\wedge J)^{-};$$

it also satisfies

$$(5) \qquad \bar{I}\vee\bar{J} = (I\vee J)^{-},$$

provided that $I\vee J$ is not a principal ideal.

PROOF. It follows immediately from (1) and (2) that (3) defines an ideal. Now, $\langle a, b\rangle \in \bar{I}\wedge\bar{J}$ iff $\langle a, b\rangle \in \bar{I}$ and $\langle a, b\rangle \in \bar{J}$, which is, by (3), equivalent to $a\in I$ and $a \in J$, that is, to $a \in I\wedge J$, which means that

$$\langle a, b\rangle \in (I\wedge J)^{-},$$

proving (4).

By (4), the inclusion "$\subseteq$" is obvious in (5). To prove the reverse inclusion, let

$$\langle a, b\rangle \in (I\vee J)^{-}.$$

Then $a \in I \vee J$ by (3); since, by hypothesis, $I \vee J$ is not principal, there exists an $a' \in I \vee J$ satisfying $a < a'$. Also, since $a' \in I \vee J$, we get elements i, j of A with $a' \leqq i \vee j$, $i \in I$, $j \in J$. By (3), $\langle i, 0\rangle \in \bar{I}$, $\langle j, 0\rangle \in \bar{J}$, and so, using (1) and (2),

$$\langle a, b\rangle < \langle i \vee j, 0\rangle = \langle i, 0\rangle \vee \langle j, 0\rangle \in \bar{I} \vee \bar{J},$$

proving the reverse inclusion, and thus Lemma 4.

Next, we need a trivial construction.

LEMMA 5. *Let K be an arbitrary lattice. K has an embedding φ into the lattice of all ideals of a suitable lattice L such that, for all $a \in K$, $a\varphi$ is a nonprincipal ideal of L.*

PROOF. For instance, let N be the chain of natural numbers, $L = K \times N$, and, for $a \in K$, set $a\varphi = \{\langle x, n\rangle \mid x \leqq a\}$.

Combining the embeddings of Lemma 4 (with, say, B the two-element chain) and Lemma 5 we obtain the main result of this note:

THEOREM 6. *Every lattice K can be embedded into the lattice of all ideals of some lattice L with no doubly reducible element.*

The Theorem of the Introduction follows immediately from Theorem 6. Indeed, if K is a transferable lattice, then we embed K into $I(L)$ by Theorem 6, where L is a lattice with no doubly reducible element. By transferability, K can be embedded into L; hence K has no doubly reducible element.

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