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METRIC AND SYMMETRIC SPACES

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ABSTRACT. In this paper we give an alternative proof, without reference to Urysohn's lemma, of the metrization theorem of Bing [2], Nagata [6], and Smirnov [8] via the theory of symmetric spaces as developed by H. Martin in [5].

A *symmetric* d on a point set X is a function $X\times X\to[0,\infty)$ satisfying (1) $d(x,y)=0$ if and only if $x=y$, and (2) $d(x,y)=d(y,x)$. A topology T on X is said to be determined by d provided that for every subset U of X, U belongs to T if and only if it contains an ε-sphere $N(p;\varepsilon)$ $(=\{x:d(p,x)<\varepsilon\})$ about each of its points p. The data X, d, and T is called a symmetric space. Such a space need not be Hausdorff or first countable, but H. W. Martin [5] has proved the theorem below.

THEOREM 1. *Let X be a topological space symmetrizable via a symmetric d. If $d(K,F)>0$ whenever $K\cap F=\varnothing$, K is compact, and F closed, then X is metrizable.*

This theorem strengthened an earlier theorem of A. V. Arhangel'skiĭ [1], who introduced the notion of symmetric spaces. Martin achieves a proof of Theorem 1 by showing that X must satisfy the hypotheses of Mrs. Frink's theorem [3], a classical result in metrization theory. As a corollary of Theorem 1, Martin (and Arhangel'skiĭ) obtains the theorem of S. Hanai and K. Morita [4], and A. H. Stone [9] on the metrizability of perfect images of metric spaces.

The purpose of this paper is to obtain the metrization theorem of Bing [2], Nagata [6], and Smirnov [8] as a consequence of Theorem 1. It is interesting to note that Urysohn's lemma is never used in this approach, as was the case in the approach used by D. Rolfsen in [7]. More specifically, let us assume that X is a regular, T_1 space with a σ-locally finite base $\mathscr{B}=\bigcup_{n=1}^{\infty}\mathscr{B}_n$, where $\mathscr{B}_n$ is locally finite and $\mathscr{B}_n\subset\mathscr{B}_{n+1}$, $n\geqq 1$.

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For $x, y \in X$, $x \neq y$, put $m(x, y) = \min\{n : \exists B \in \mathscr{B}_n$ with $x \in B$, $y \notin \bar{B}\}$, $t(x, y) = 1/m(x, y)$, and $d(x, y) = \max\{t(x, y), t(y, x)\}$. Also, put $d(x, x) = 0$. Then we shall prove the following theorem.

THEOREM 2. *X is symmetrizable via d. Furthermore, $d(K, F) > 0$ whenever $K \cap F = \varnothing$, K is compact, and F closed. Therefore, X is metrizable.*

PROOF. Denote by T and T_d the given and d-induced topologies on X, respectively. We must show that (1) $T \subset T_d$, (2) $T_d \subset T$, and (3) $d(K, F) > 0$ whenever $K \cap F = \varnothing$, K is compact, and F closed.

To establish (1), assume that $B \in \mathscr{B}$, $x \in B$. Choose $B_1 \in \mathscr{B}$ such that $x \in B_1 \subset \bar{B}_1 \subset B$. If $B_1 \in \mathscr{B}_n$, we have $N(x; 1/n) \subset \bar{B}_1 \subset B$, so that B is open in T_d.

To establish (2), let F be a T_d-closed set. If F is not T-closed (X is first countable because of σ-locally finite $\mathscr{B}$), there is a point $x \notin F$ and a sequence $x_1, x_2, \cdots$ of points in F converging to x. We shall show that

(i) $\lim_{i \to \infty} t(x, x_i) = 0$,

(ii) $\inf\{t(x_i, x) : i \geqq 1\} = 0$, so that

(iii) $\inf\{d(x, x_i) : i \geqq 1\} = 0$ holds, which contradicts $d(x, F) > 0$.

To this end, let $x \in B \in \mathscr{B}_n$. Denote by U the intersection of all members of $\mathscr{B}_n$ containing x. There exists a positive integer N satisfying $x_i \in U$ for $i \geqq N$, whence $t(x, x_i) < 1/n$. Since n can be chosen as large as we please, (i) follows.

As for (ii), let $x \in B \in \mathscr{B}_n$. Denote by V an open neighborhood of x that intersects only finitely many members of $\mathscr{B}_n$ and satisfies $V \subset B$. Choose N so that $x_i \in V$ for $i \geqq N$. Whenever $i \geqq N$, let U_i represent the intersection of all members of $\mathscr{B}_n$ containing x_i. It follows that for infinitely many such values of i, the sets U_i are identical, there being only finitely many such intersections. Denoting such a common value by U, it is clear that $x \in \bar{U}$, and therefore that $t(x_i, x) < 1/n$. This establishes (ii), (iii), and (2).

To establish (3), let K be compact, F closed, and $K \cap F = \varnothing$. Let $B_1, B_2, \cdots, B_k$ be a finite cover of K by members of $\mathscr{B}$ with $\bar{B}_i \cap F = \varnothing$, $i = 1, \cdots, k$. Choose n such that $B_i \in \mathscr{B}_n$, $i = 1, \cdots, k$. Then we have $0 < 1/n \leqq t(K, F) \leqq d(K, F)$.

REFERENCES

1. A. V. Arhangel′skiĭ, *Mappings and spaces*, Uspehi Mat. Nauk **21** (1966), no. 4 (130), 133–184 = Russian Math. Surveys **21** (1966), no. 4, 115–162. MR **37** #3534.

2. R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. **3** (1951), 175–186. MR **13**, 264.

3. A. H. Frink, *Distance functions and the metrization problem*, Bull. Amer. Math. Soc. **43** (1937), 133–142.

4. S. Hanai and K. Morita, *Closed mappings and metric spaces*, Proc. Japan. Acad. **32** (1956), 10–14. MR **19**, 299.

5. H. W. Martin, *Metrization of symmetric spaces and regular maps*, Proc. Amer. Math. Soc. **35** (1972), 269–274.

6. J. Nagata, *On a necessary and sufficient condition for metrizability*, J. Inst. Polytech. Osaka City Univ. Ser. A Math. **1** (1950), 93–100. MR **13,** 264.

7. D. Rolfsen, *Alternative metrization proofs*, Canad. J. Math. **18** (1966), 750–757. MR **33** #6585.

8. Ju. M. Smirnov, *A necessary and sufficient condition for metrizability of a topological space*, Dokl. Akad. Nauk SSSR (N.S.) **77** (1951), 197–200. (Russian) MR **12,** 845.

9. A. H. Stone, *Metrizability of decomposition spaces*, Proc. Amer. Math. Soc. **7** (1956), 690–700. MR **19,** 299.

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