HOLOMORPHIC FIBER BUNDLES WITH HYPERBOLIC FIBER¹

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ABSTRACT. The purpose of the present note is to show that holomorphic fiber bundles with hyperbolic fiber are almost trivial: They are trivial if the base is simply connected, and have a simple structure otherwise.

Kobayashi [2], [3] has introduced the notion of a hyperbolic complex manifold. We need not be concerned with the definition here, but only with the following consequence [3], [4]: If M is a hyperbolic complex manifold, p a point of M , and U a neighborhood of p , then there is a $\delta > 0$ such that any holomorphic map ϕ of the unit disk into M with $\phi(0) = p$ maps the disk $\{z: |z| < \delta\}$ into U . From this, it follows that, if $w^1, \dots, w^n$ are coordinates at p , and l an integer, then there are constants A_l depending only on l and the choice of coordinates such that, if ϕ is any map of the unit disk into M with $\phi(0) = p$, then $|\partial^l w^i / \partial z^l| < A_l$ at $z = 0$.

The crux of our argument is the following proposition, which generalizes a result of Kobayashi's that a complex Lie group (of dimension greater than 0) cannot act holomorphically on a hyperbolic manifold.

PROPOSITION. *Let M and N be complex manifolds with M hyperbolic. Let $F = F(w, z)$ be a holomorphic map of $M \times N$ into M . Suppose that, for some $z_0 \in N$, the restriction of F to $M \times \{z_0\}$ is a biholomorphic map onto M . Then F is independent of z ; i.e., $F(w, z) = F(w, z_0)$.*

PROOF. It suffices to prove the proposition for the case that N is a ball, and hence for the case that N is a disk Δ , which we take to be the unit disk with $z_0 = 0$. Without loss of generality, we may assume $F(w, z_0) = w$. Near a point $\langle w_0, z_0 \rangle$, the map F has a power series expansion of the form

$$F^i(w, z) = w^i + a^i(w)z^l + O(z^{l+1}).$$

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We inductively define maps ϕ_k of the unit disk into M by setting $\phi_0 = w_0$ and $\phi_k(z) = F(\phi_{k-1}(z), z)$. Then near $z=0$, the map ϕ_k has the expansion

$$\phi_k^i(z) = w_0^i + ka^i(w_0)z^l + O(z^{l+1}).$$

Since M is hyperbolic, there is a constant A_l independent of k such that

$$l! |ka^i(w_0)| = \left| \frac{\partial^l \phi_k^i}{\partial z^l} \right| < A_l.$$

Hence $a^i(w_0) = 0$. But this means that the derivatives of F of all orders with respect to z vanish at $\langle w_0, z_0 \rangle$. Since w_0 was an arbitrary point of M , we must have F independent of z , proving the proposition.

As an immediate consequence, we have the following two corollaries:

COROLLARY 1. *A holomorphic fiber bundle with simply connected base and hyperbolic fiber is trivial.*

COROLLARY 2. *The holomorphic fiber bundles with hyperbolic fiber M and base B are in natural one-to-one correspondence with the homomorphisms of the fundamental group of B into the group of biholomorphic automorphisms of M onto itself.*

If we make use of the facts that the product of two (complete) hyperbolic manifolds is (complete) hyperbolic and that a manifold is (complete) hyperbolic iff its universal covering manifold is [3], [4], we obtain from Corollary 1 the following result of Kiernan [1]:

COROLLARY 3. *A holomorphic fiber bundle with (complete) hyperbolic fiber and base is (complete) hyperbolic.*

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