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ON A WEAKLY CLOSED SUBSET OF THE SPACE OF τ-SMOOTH MEASURES

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ABSTRACT. It is known that a lot of topological properties devolve from a basic space X to the family $M_\tau(X)$ of all τ-smooth Borel measures endowed with the weak topology (or to certain subspaces of $M_\tau(X)$). The aim of this paper is to show that among these topological properties there cannot be properties which are hereditary on closed subsets but not on countable products of X, e.g. normality, paracompactness, the Lindelöf property, local compactness and σ-compactness. For this purpose it is proved that the countable product space X^N is homeomorphic to a closed subset of $M_\tau(X)$. A further consequence of this result is for example that, for the family $M_\tau^1(X)$ of probability measures in $M_\tau(X)$, compactness, local compactness and σ-compactness are equivalent properties.

Let X be a Hausdorff space. By $M(X)$ we denote the family of all nonnegative finite-valued measures defined on the Borel field of X. Let $M(X)$ be endowed with the *weak topology*, i.e. the weakest topology for which each map $\mu\to\mu G$, where G is an open subset of X, is lower semicontinuous and $\mu\to\mu X$ is continuous.

It is an immediate consequence of the definition of this topology that a net $(\mu_\alpha)_{\alpha\in D}$ of measures converges (weakly) to a measure μ in $M(X)$ if and only if $\mu X=\lim \mu_\alpha X$ and either $\mu G\leqq\liminf \mu_\alpha G$ for every open set $G\subset X$ or $\mu F\geqq\limsup \mu_\alpha F$ for every closed set $F\subset X$. Further criteria for weak convergence of measures may be gathered from [3, Theorem 8.1].

Two subsets of $M(X)$, the families $M_t(X)$ of tight and $M_\tau(X)$ of τ-smooth measures in $M(X)$, are of special interest. A measure $\mu\in M(X)$ is said to be *tight* if

$$\mu A = \sup\{\mu K \mid K\subset A, K \text{ compact}\}$$

for each Borel set A and *τ-smooth* if

$$\mu F_0 = \inf\{\mu F \mid F\in\mathscr{F}\}$$

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for each system $\mathscr{F}$ of closed subsets of X directed downwards to the closed set F_0. Clearly, $M_t(X)$ is a subset of $M_\tau(X)$. The sets of all probability measures in $M_t(X)$ and $M_\tau(X)$ are denoted by $M_t^1(X)$ and $M_\tau^1(X)$.

Let ε_x be the probability measure giving mass unity to the point $x \in X$. Then $\sum_{n=1}^\infty c_n\varepsilon_{x_n}$ with $c_n \geqq 0$, $n \in N$, is the measure with mass c_n in the point $x_n \in X$, $n \in N$.

If A is a subset of X, we write A° for the interior, A^c for the closure and 1_A for the characteristic function of A.

LEMMA 1. *Let $c_n \geqq 0$, $n \in N$, be a sequence of real numbers such that $\sum_{n=1}^\infty c_n < \infty$. Then $(x_n)_{n\in N} \rightarrow \sum_{n=1}^\infty c_n\varepsilon_{x_n}$ defines a continuous mapping from X^N into $M_t(X)$.*

PROOF. Let $((x_{n,\alpha})_{n\in N})_{\alpha\in D}$ be a net in X^N and $(x_n)_{n\in N} \in X^N$ such that $x_{n,\alpha} \rightarrow x_n$ $(\alpha \in D)$ for each $n \in N$. Put

$$\mu_\alpha = \sum_{n=1}^\infty c_n\varepsilon_{x_{n,\alpha}} \quad (\alpha \in D) \quad \text{and} \quad \mu = \sum_{n=1}^\infty c_n\varepsilon_{x_n}.$$

Obviously, the map $x \rightarrow \varepsilon_x$ is continuous; hence $\varepsilon_{x_{n,\alpha}} \rightarrow \varepsilon_{x_n}$ $(\alpha \in D)$ for each $n \in N$. Let $\varepsilon > 0$ and $G \subset X$ be an open set. Then there is $m \in N$ such that

$$\begin{aligned}\mu G - \varepsilon &\leqq \sum_{n=1}^m c_n\varepsilon_{x_n}(G) \leqq \sum_{n=1}^m c_n \liminf_\alpha \varepsilon_{x_{n,\alpha}}(G)\\ &\leqq \liminf_\alpha \sum_{n=1}^m c_n\varepsilon_{x_{n,\alpha}}(G) \leqq \liminf_\alpha \mu_\alpha G.\end{aligned}$$

This implies $\mu G \leqq \liminf \mu_\alpha G$ for every open set G. Hence $\mu_\alpha \rightarrow \mu$.

THEOREM 1. *Let $c_n \geqq 0$, $n \in N$, be a sequence of real numbers such that $\sum_{n=1}^\infty c_n < \infty$. Then $\{\mu \mid \mu = \sum_{n=1}^\infty c_n\varepsilon_{x_n}, x_n \in X\ (n \in N)\}$ is a closed subset of $M_\tau(X)$.*

PROOF. Let $\mu \in M_\tau(X)$ and $((x_{n,\alpha})_{n\in N})_{\alpha\in D}$ be a net in X^N such that $\mu_\alpha \rightarrow \mu$, where $\mu_\alpha = \sum_{n=1}^\infty c_n\varepsilon_{x_{n,\alpha}}$ $(\alpha \in D)$. Without loss of generality we may assume that the net $(x_{n,\alpha})_{\alpha\in D}$ is constant for each $n \in N$ with $c_n = 0$. Further let $(\alpha_\beta)_{\beta\in E}$ be a universal subnet of D (more precisely, of the identity map of D). Then $(x_{n,\alpha_\beta})_{\beta\in E}$ is a universal subnet of $(x_{n,\alpha})_{\alpha\in D}$ for each $n \in N$.

Let $n \in N$ such that $c_n > 0$ and assume that for each $x \in X$ there is a neighborhood $N(x)$ of x such that x_{n,α_β} is eventually lying in the complement of $N(x)$. Then the system

$$\mathscr{F} = \{F \mid F \subset X \text{ closed}, x_{n,\alpha_\beta} \text{ eventually in } F\}$$

is directed downwards to the empty set. Since μ is τ-smooth, it follows that $\inf\{\mu F | F \in \mathscr{F}\}=0$. This leads to a contradiction, as on the other hand $\mu_{\alpha_\beta} \to \mu$, and hence

$$\mu F \geqq \limsup_\beta \mu_{\alpha_\beta} F \geqq \limsup_\beta c_n 1_F(x_{n,\alpha_\beta}) = c_n > 0$$

for every $F \in \mathscr{F}$.

Since $(x_{n,\alpha_\beta})_{\beta \in E}$ is universal, this implies that for each $n \in N$ there is $x_n \in X$ such that $x_{n,\alpha_\beta} \to x_n$ $(\beta \in E)$. By Lemma 1 we may conclude that

$$\mu_{\alpha_\beta} \to \sum_{n=1}^{\infty} c_n \varepsilon_{x_n} =: \mu' \qquad (\beta \in E).$$

We claim that $\mu=\mu'$. Since $\mu X = \lim_\beta \mu_{\alpha_\beta} X = \mu' X$, it suffices to prove that $\mu\{x_n\}=\mu'\{x_n\}$ for every $n \in N$. Let $n \in N$ and $\varepsilon>0$ be given. Because X is Hausdorff, the system of all closed subsets F of X for which x_n lies in the interior F° of F is directed downwards to the set $\{x_n\}$. Hence, as μ is τ-smooth, there exists a closed set $F \subset X$ such that $x_n \in F^\circ$ and $\mu F \leqq \mu\{x_n\}+\varepsilon$, and it follows that

$$\begin{aligned}\mu'\{x_n\} &\leqq \mu' F^\circ \leqq \liminf_\beta \mu_{\alpha_\beta} F^\circ \leqq \limsup_\beta \mu_{\alpha_\beta} F \\ &\leqq \mu F \leqq \mu\{x_n\} + \varepsilon.\end{aligned}$$

Thus $\mu'\{x_n\} \leqq \mu\{x_n\}$. The inverse inequality is proved analogously, since μ' is τ-smooth, too.

As a special case the preceding theorem includes the result that the collection of one-point probability measures on X is a closed subset of $M_\tau(X)$. This set is however not necessarily closed in $M(X)$, as the following example shows.

Let X be the space of all ordinals up to and including the first uncountable ordinal Ω. With the usual order topology X is a compact Hausdorff space. It is well known that the set function μ defined by $\mu A=1$ or $\mu A=0$ according as the Borel set A does or does not contain an unbounded closed subset of the space $X-\{\Omega\}$ is a measure, i.e. $\mu \in M(X)$ (cf. **[1**, p. 231, (10)**]**). Obviously this measure μ is the weak limit of the net $(\varepsilon_\alpha)_{\alpha<\Omega}$.

Note that both μ and ε_Ω are limit measures of the net $(\varepsilon_\alpha)_{\alpha<\Omega}$. This shows that even for a compact Hausdorff space X the space $M(X)$ need not be Hausdorff, too.

THEOREM 2. *Let $c_n>0$, $n \in N$, be a sequence of real numbers such that $\sum_{n=m+1}^{\infty} c_n < c_m$ for each $m \in N$. Then $(x_n)_{n\in N} \to \sum_{n=1}^{\infty} c_n \varepsilon_{x_n}$ defines a homeomorphism T from X^N into $M_t(X)$.*

PROOF. Since $c_m > \sum_{n=m+1}^{\infty} c_n$ for each $m \in N$, it is quite evident that $\sum_{n \in I_1} c_n = \sum_{n \in I_2} c_n$ implies $I_1 = I_2$ for all $I_1, I_2 \subset N$ and therefore T is one-to-one.

We now claim that T^{-1} is continuous. Let $((x_{n,\alpha})_{n \in N})_{\alpha \in D}$ be a net in X^N and $(x_n)_{n \in N} \in X^N$. Suppose $\mu_\alpha \to \mu$, where

$$\mu_\alpha = \sum_{n=1}^{\infty} c_n \varepsilon_{x_{n,\alpha}} \quad (\alpha \in D) \quad \text{and} \quad \mu = \sum_{n=1}^{\infty} c_n \varepsilon_{x_n}.$$

To prove that $x_{n,\alpha} \to x_n$ $(\alpha \in D)$ for each $n \in N$, we proceed by induction.

Let U be some open neighbourhood of x_1. Since, for each $\alpha \in D$,

$$\mu_\alpha U = \sum_{n=1}^{\infty} c_n 1_U(x_{n,\alpha}) \leqq c_1 1_U(x_{1,\alpha}) + \sum_{n=2}^{\infty} c_n,$$

it follows that

$$\begin{aligned} c_1 \leqq \mu U \leqq \liminf_\alpha \mu_\alpha U &\leqq \sum_{n=2}^{\infty} c_n + \liminf_\alpha c_1 1_U(x_{1,\alpha}) \\ &< c_1 + \liminf_\alpha c_1 1_U(x_{1,\alpha}). \end{aligned}$$

Thus $\liminf_\alpha c_1 1_U(x_{1,\alpha}) > 0$, which implies that $x_{1,\alpha} \in U$, eventually. Hence $x_{1,\alpha} \to x_1$.

Assume now that $x_{n,\alpha} \to x_n$ $(\alpha \in D)$ for each $n \in \{1, \cdots, m-1\}$ with some $m > 1$. Define $I = \{n \mid n < m,\ x_n = x_m\}$, $J = \{n \mid n < m,\ x_n \neq x_m\}$ and let U be an open neighborhood of x_m of which we may and do assume that $\{x_n \mid n \in J\} \cap U^c = \varnothing$. Then there exists $\alpha_0 \in D$ such that for each $\alpha \geqq \alpha_0$ we have $x_{n,\alpha} \in U$ if $n \in I$ and $x_{n,\alpha} \in X - U$ if $n \in J$. Thus

$$\mu_\alpha U = \sum_{n \in I} c_n + \sum_{n=m}^{\infty} c_n 1_U(x_{n,\alpha}) \leqq \sum_{n \in I} c_n + \sum_{n > m} c_n + c_m 1_U(x_{m,\alpha})$$

for every $\alpha \geqq \alpha_0$, and we conclude that

$$\begin{aligned} \sum_{n \in I} c_n + c_m \leqq \mu U &\leqq \liminf_\alpha \mu_\alpha U \\ &\leqq \sum_{n \in I} c_n + \sum_{n > m} c_n + \liminf_\alpha c_m 1_U(x_{m,\alpha}) \\ &< \sum_{n \in I} c_n + c_m + \liminf_\alpha c_m 1_U(x_{m,\alpha}). \end{aligned}$$

Hence $\liminf_\alpha c_m 1_U(x_{m,\alpha}) > 0$ for every neighborhood U of x_m, whence $x_{m,\alpha} \to x_m$ $(\alpha \in D)$.

Since the map T is continuous by Lemma 1, it has the required properties.

COROLLARY. *The countable product space X^N is homeomorphic to a closed subset of $M_\tau(X)$ consisting only of tight probability measures.*

This corollary yields that a topological property which is hereditary on closed subsets cannot devolve from X onto $M_\tau(X)$, $M_\tau^1(X)$, $M_t(X)$ or $M_t^1(X)$, respectively, unless it devolves from X to X^N. Hence we obtain the result that for a normal space X the weak topology of these spaces is not normal in general, since even $X \times X$ may already fail to be normal (cf. [**2**, p. 133, 4.I]). The same holds for paracompact and for Lindelöf spaces (cf. [**2**, p. 113, Lemma 1 and p. 159, Corollary 32 in addition to 4.I]).

Furthermore we know that X has to be compact in case X^N is locally compact or σ-compact (i.e. the union of countably many compact sets). Thus, if $M_t^1(X)$ or $M_\tau^1(X)$ is locally compact or σ-compact, then X is compact, in which case $M_t^1(X)$ and $M_\tau^1(X)$ are compact, too (see Theorems 9.1 (iii) and 9.2 of [**3**]), i.e. the weak topology of $M_t^1(X)$ and $M_\tau^1(X)$ is necessarily compact, if it is locally compact or σ-compact.

We may also conclude (by the same theorems of [**3**]) that $M_\tau(X)$, as well as $M_t(X)$, is locally compact, respectively σ-compact, if and only if X is compact, since the sets $\{\mu | \mu X \leqq n\}$, $n \in N$, are compact if X is compact, and each μ is contained in the interior of one of these sets.

Finally we like to refer the reader who wishes to have information about those topological properties that devolve from a basic space X to spaces of measures on X (in their weak topology) to the papers of Varadarajan [**4**] and Topsøe [**3**, Theorem 11.2].

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