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ON ISOMORPHIC GROUPS AND HOMEOMORPHIC SPACES

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ABSTRACT. Let $C(X, G)$ denote the group of continuous functions from a topological space X into a topological group G with the pointwise multiplication. Some classes of SQ-pairs and properties of the corresponding topological group $C(X, G)$ with the compact-open topology are investigated. We also show that the existence of a group isomorphism between groups $C(X, G)$ and $C(Y, G)$ implies the existence of a homeomorphism between X and Y, if (X, G) and (Y, G) are SQ-pairs.

1. **Introduction.** For a topological space X and a topological group G, let $C(X, G)$ be the group of all continuous functions from X into G with the pointwise multiplication, that is, $(fg)(x)=f(x)g(x)$; the identity element of the group $C(X, G)$ is the constant function $I_0(X, G)$, or simply I_0, which maps every x in X into the identity element e of G. It is well known that if $C(X, G)$ is endowed with the compact-open topology, it becomes a topological group. It is clear that if h is a homeomorphism of X onto Y, then $f \to f \circ h$ is an isomorphism from $C(Y, G)$ onto $C(X, G)$ which maps every constant function on Y into the corresponding constant function on X. We are concerned, in this paper, with the question: If a group isomorphism exists between $C(Y, G)$ and $C(X, G)$ which maps every constant function on Y into the corresponding constant function on X, does there exist a homeomorphism between X and Y? In general, the answer to this question is, of course, no, for we may take X to be a noncompact pseudocompact space, and then there is a ring isomorphism between the rings $C(X, R)$ and $C(\beta X, R)$ but X and βX are not homeomorphic.

We find that the answer to the above question is yes for certain pairs (X, G) of topological space X and topological group G. Such pairs are termed SQ-pairs as defined in [**9**]. §3 is devoted to proving this assertion by showing first that, if X is a k-space, X is homeomorphic to the space of all c-continuous homomorphisms of the topological group $C(X, G)$ onto the topological group G with F-normal subgroups as kernels and

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endowed with the compact-open topology. We disclose some classes of SQ-pairs, and some properties of $C(X, G)$ in §2.

All topological spaces considered here are assumed to be Hausdorff.

2. **SQ-pairs.** For each p in X, let $M_p=\{f\in C(X,G):f(p)=e\}$, and let h_p be the evaluation map of $C(X, G)$ onto G defined by $h_p(f)=f(p)$. For each r in G, let $\mathbf{r}$ denote the constant function in $C(X, G)$ which maps X into r. Then h_p is a continuous homomorphism of $C(X, G)$ onto G with M_p as its kernel and maps every constant function $\mathbf{r}$ into r. Hence we see that $C(X, G)/M_p$ is isomorphic to G under the continuous isomorphism that maps every coset $\mathbf{c}M_p$ into c. Note that for each p in X, every coset $\mathbf{c}M_p$ contains exactly one constant map, namely $\mathbf{c}$. For the sake of convenience, let us call a homomorphism of $C(X, G)$ (or $C(X, G)/M$) onto G a c-homomorphism if it maps every $\mathbf{r}$ (resp. $\mathbf{r}M$) into r. Every evaluation map is a c-continuous homomorphism of $C(X, G)$ onto G.

In contrast to the fact that every nonzero homomorphism of $C(X)=C(X, R)$ onto R is a c-homomorphism [**5**, 10.5], not every continuous homomorphism of $C(X, G)$ onto G is a c-continuous homomorphism, as the following example shows.

Example. Let G be the additive group of integers modulo 2 with the discrete topology. Then $C(G, G)=\{I_0, f_1, f_2, f_3\}$, where f_1 is the function which maps G into 1, f_2 is the function which maps 1 into 1 and 0 into 0, and f_3 is the one which maps 0 into 1 and 1 into 0. The compact-open topology for $C(G, G)$ is the discrete topology. If we define a mapping h: $C(G, G)\to G$ by defining $h(I_0)=h(f_1)=0$, $h(f_2)=h(f_3)=1$, then h is an onto homomorphism, yet it is not a c-homomorphism.

For $f\in C(X, G)$, we let $Z(f)=\{x\in X:f(x)=e\}$, and for a subgroup M of $C(X, G)$, let $Z(M)=\{Z(f):f\in M\}$. Note that, for any f and g in $C(X, G)$,

$$Z(fg)\supset Z(f)\cap Z(g),\qquad Z(f^{-1})=Z(f)\quad\text{and}\quad Z(fgf^{-1})=Z(g).$$

Definition 1 [**9**]. We shall call a pair (X, G) of a topological space X and a topological group G an S-pair if, for each closed subset C of X and $x\notin C$, there exists $f\in C(X, G)$ such that $Z(f)\supset C$ and $f(x)\neq e$.

It is clear that (X, R) is an S-pair for every completely regular space, and that if (X, G) is an S-pair then X is completely regular.

Remark 1. If X is a topological space such that each x in X has a local base U_x satisfying the property that, for each U in U_x there exists a continuous function f of $\bar{U}$ into G such that $f(x)\neq e$ but $f(y)=e$ for each y in $\bar{U}-U$, then (X, G) is an S-pair. To see this, let C be a closed subset of X and $x\notin C$. Then, for some U in U_x, $x\in U\subset X-C$; and let f be a continuous function on $\bar{U}$ into G such that $f(x)\neq e$ but $f(y)=e$ for each y

in $\bar{U}-U$. Define $g:X\to G$ such that $g=f$ on $\bar{U}$ and $g(y)=e$ for $y\notin \bar{U}$. Then $g\in C(X, G)$, $Z(g)\supset C$, and $g(x)\neq e$.

REMARK 2. If X is completely regular, and G is path connected, then (X, G) is an S-pair. To see this let $t\neq e$ be in G. If C is a closed subset of X and $x\notin C$, let f be a continuous function of X into $[0, 1]$ such that $f(x)=1$ and $f(C)=\{0\}$, and let $g:[0, 1]\to G$ be the path such that $g(0)=e$ and $g(1)=t$. Then $g\circ f$ is the desired function in $C(X, G)$.

REMARK 3. For every zero-dimensional space X, (X, G) is an S-pair.

We point out that, if B is a closed subset of X and (X, G) is an S-pair, then (B, G) is also an S-pair.

DEFINITION 2 [**9**]. (1) A normal subgroup M of $C(X, G)$ is called an F-normal subgroup if $\{Z(f):f\in M\}$ has the finite intersection property.

(2) A pair (X, G) of a topological space X and a topological group G is called a Q-pair if whenever M is an F-normal subgroup of $C(X, G)$ such that $C(X, G)/M$ is isomorphic to G by a c-isomorphism, then $\bigcap Z(M)\neq\varnothing$.

It is clear that if X is a completely regular space such that (X, R) is a Q-pair, then X is realcompact. As pointed out in [**9**], (X, G) is a Q-pair if X can be embedded into G as a subspace of G. Since every completely regular space X is a closed subspace of the free topological group $F(X)$ generated by X, and every topological group can be embedded as a closed subgroup of a path connected and locally path connected topological group [**6**], we see that for every completely regular space X there exists a path connected and locally path connected topological group G such that (X, G) is an SQ-pair. If X is compact, (X, R) is an SQ-pair.

If (X, G) is a Q-pair, then the only F-normal subgroups of $C(X, G)$ such that $C(X, G)/M$ is c-isomorphic to G are of the form M_p, $p\in X$ [**9**]. Thus we have the following:

PROPOSITION 4. *An S-pair (X, G) is a Q-pair if and only if every c-homomorphism h of $C(X, G)$ onto G with an F-normal subgroup as its kernel is of the form h_p for some $p\in X$.*

PROOF. For the necessity, let M be the kernel of h, then $C(X, G)/M$ is c-isomorphic to G. Hence there is $p\in\bigcap Z(M)$ such that $M=M_p$. Therefore $\ker h=\ker h_p$. Now for $f\in C(X, G)$, let $f(p)=c$, and let $g=fc^{-1}$, then $g\in M_p=M$. Hence

$$h(f) = h(gc) = h(g)h(c) = h(g)c = c = f(p) = h_p(f).$$

This shows that $h=h_p$.

For the sufficiency, suppose M is an F-normal subgroup of $C(X, G)$ such that $C(X, G)/M$ is c-isomorphic to G by the c-isomorphism k. Let $h=k\circ\alpha$, where α is the natural map of $C(X, G)$ onto $C(X, G)/M$. Then h

is a c-homomorphism of $C(X, G)$ onto G with M as its kernel. Hence there is a unique $p \in X$ such that $h=h_p$, and thus $M=M_p$.

Following [7], we call a topological space X a V-space if for points p, q, x, and y of X, where $p \neq q$, there exists a continuous function f of X into itself such that $f(p)=x$ and $f(q)=y$. It is shown in [7] that every completely regular path connected space and every zero-dimensional space is a V-space.

Recall that a topological space X is said to be an S-space if, for each pair of distinct points of X, there is a continuous real-valued function on X whose values at these points do not coincide. R. Arens defined it in [**1**], and has shown that, if the space $C(X, R)$ satisfies the first axiom of countability and X is an S-space, then X is hemicompact. Adopting the same line of argument, we have the following:

THEOREM 5. *If (X, G) is an S-pair, G is a V-space, and if $C(X, G)$ satisfies the first axiom of countability, then X is hemicompact and G is metrizable.*

PROOF. Since G can be embedded as a retract of $C(X, G)$, G is metrizable. For the hemicompactness of X, the proof is not different from that of [**1**, Theorem 8] and thus omitted.

It is remarked that, if $X=\bigcup_{n=1}^{\infty} C_n$ where $C_1 \subset C_2 \subset C_3, \cdots$, is hemicompact and if $\{V_n\}$ is a countable local base for e in G, then $\{(C_n, V_m)\}$ is a local base at I_0 in $C(X, G)$, and hence $C(X, G)$ is metrizable, where $(C_n, V_m)=\{f \in C(X, G): f(C_n) \subset V_m\}$.

LEMMA 6. *Let (X, G) be an S-pair, and let Ω be an open covering for X. For each closed subset C of X contained in some member of Ω and for each open neighborhood U of e in G, let $(C, U)=\{f \in C(X, G): f(C) \subset U\}$. Then the topology t for the group $C(X, G)$ having the family of sets of the form (C, U) as subbasic neighborhoods of I_0 is jointly continuous, that is, the map $P: X \times C(X, G) \rightarrow G$ defined by $P(f, x)=f(x)$ is continuous.*

PROOF. Let $f \in C(X, G)$, $x \in X$, and let W be a neighborhood of $f(x)$. Then $f(x)U \subset W$ for some open set U in G containing e, and hence $x \in f^{-1}(f(x)V) \cap O$, where $x \in O \in \Omega$ and V an open neighborhood of e such that $V^2 \subset U$. If C is a closed neighborhood of x such that $C \subset f^{-1}(f(x)V) \cap O$, then, for $g \in f(C, V)$ and $y \in C$, $g(y) \in f(y)V \subset f(x)U \subset W$. Hence P is continuous.

THEOREM 7. *Let (X, G) be an S-pair, where G is a V-space. If there exists a smallest jointly continuous topology t for the group $C(X, G)$, then X is locally compact.*

PROOF. The proof is similar to that of [**1**, Theorem 3]. Let a be an element of G different from e, and let U be a neighborhood of e in G such that $a \notin U$, and let $x \in X$. By the joint continuity of t, let V be a neighborhood of x, and W a t-neighborhood of I_0 such that $g(V) \subset U$ for every g in W. We want to show that $\bar{V}$ is compact.

Let Ω be an open covering for $\bar{V}$, and let $\Omega' = \{X - \bar{V}\} \cup \Omega$. Then Ω' is an open covering for X. Let t' be the topology for $C(X, G)$ induced by Ω' as described in Lemma 6, then we have $t \subset t'$. Hence there are closed sets $C_i \subset O_i$ of X and open neighborhoods U_i of e in G, $i=1, 2, \cdots, n$, such that $W' = \bigcap_{i=1}^{n} (C_i, U_i)$ is contained in W. Let $O = V - \bigcup_{i=1}^{n} C_i$, and suppose that $p \in O$. Then there is f in $C(X, G)$ such that $Z(f) \supset X - O$ and $f(p) \neq e$. Let g be a continuous function of G into itself with $g(e) = e$ and $g(f(p)) = a$, and let $g = g \circ f$. Then $h(X - O) = e$ and $h(p) = a$, hence $h \in W'$. But p is in V and $h(p) = a \notin U$; we have $h \notin W$ which is impossible. Hence $O = \varnothing$, and we have $\bar{V} \subset \bigcup_{i=1}^{n} C_i \subset \bigcup_{i=1}^{n} O_i$. Therefore $\bar{V}$ is compact.

COROLLARY. *If (X, G) is an S-pair, where G is a V-space, and $X \times C(X, G)$ is a k-space, where $C(X, G)$ has the compact-open topology, then X is locally compact.*

PROOF. If $X \times C(X, G)$ is a k-space, then the compact-open topology for $C(X, G)$ is jointly continuous [**2**]; hence X is locally compact.

The above corollary generalizes a result in [**2**]. As an application, we show in the following example that the product of two topological groups which are k-spaces need not be a k-space, a fact pointed out by N. Noble [**8**].

EXAMPLE. Let X be the dual space of an infinite-dimensional Fréchet space with the compact-open topology. Then X is a topological group which is a hemicompact k-space but is not locally compact. If G is any metrizable topological group which is also a V-space such that (X, G) is an S-pair, then $C(X, G)$ is metrizable by the remark following Theorem 5. Since X is not locally compact, $X \times C(X, G)$ is a topological group but is not a k-space as follows from the above corollary. This example was cited by N. Noble [**8**] for the case where G is the additive group of real numbers.

3. **Isomorphic groups.** This section is devoted to prove the following:

THEOREM 8. *Suppose that (X, G) and (Y, G) are SQ-pairs. If there exists an isomorphism between groups $C(Y, G)$ and $C(X, G)$ which maps every constant function on Y into the corresponding constant function on X, then X and Y are homeomorphic.*

All pairs (Z, G) considered in this section are assumed to be SQ-pairs. Since every noncompact pseudocompact space X is not realcompact,

(X, R) cannot be a Q-pair, thus Theorem 8 is false if (X, G) is not a Q-pair.

In order to establish Theorem 8, we first prove that, if X is a k-space, X is homeomorphic to the space of all c-continuous homomorphisms of the topological group $C(X, G)$ onto the topological group G with F-normal subgroups as kernels and endowed with the compact-open topology; let $H(X, G)$ denote such a space of c-continuous homomorphisms. For each $p \in X$, the evaluation map h_p is in $H(X, G)$, hence the correspondence $p \to h_p$ defines a map μ from X into $H(X, G)$.

THEOREM 9. *If X is a k-space, the mapping μ is a homeomorphism of X onto $H(X, G)$.*

PROOF. Proposition 4 implies that μ is onto.

If $p \neq q$ in X, there is $f \in C(X, G)$ such that $f(p) \neq f(q)$, hence $h_p(f) \neq h_q(f)$. Thus μ is one-to-one.

The continuity of μ follows from Theorem 2 of [**4**], which states that if F is a family of continuous functions from a k-space X into a regular space Y endowed with the compact-open topology, then the mapping $\theta: X \to C(F, Y)$ defined by $\theta(x)(f) = f(x)$ is continuous, where $C(F, Y)$ also has the compact-open topology.

It remains to show that μ is a closed map. Let C be a closed subset of X. Then $\mu(C) = \{h_x : x \in C\}$. Let $\{h_{x_n}\}_{n \in A}$ be a net in $\mu(C)$ such that $h_{x_n} \to h_x$ in $H(X, G)$, where $x_n \in C$ for each $n \in A$. If $x \notin C$, then there exists an f in $C(X, G)$ such that $f(x) \notin \mathrm{cl}[f(C)]$. But $h_{x_n}(f) \to h_x(f)$ in G; we have $f(x_n) \to f(x)$ in G, hence $f(x) \in \mathrm{cl}[f(C)]$, a contradiction. Hence $x \in C$ and $\mu(C)$ is closed.

REMARK 10. The hypothesis that X is a k-space in Theorem 9 is merely to assure the continuity of μ. In fact, if $H(X, G)$ is given the point-open topology instead of the compact-open topology, the mapping μ is easily seen to be continuous without assuming that X is a k-space.

Suppose now that $\theta: X \to Y$ is a continuous map of a k-space X into a k-space Y. Define $\theta': C(Y, G) \to C(X, G)$ by setting $\theta'(g) = g \circ \theta$ for each g in $C(Y, G)$ into the corresponding constant function in $C(X, G)$. Note that if $h_x \in H(X, G)$, then $h_x \circ \theta'$ is in $H(Y, G)$. Hence we have a continuous mapping θ'' of $H(X, G)$ onto $H(Y, G)$ defined by $\theta''(h_x) = h_x \circ \theta'$ for each $h_x \in H(X, G)$. It is easy to verify that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\theta} & Y \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle \mu_Y} \\ H(X, G) & \xrightarrow[\theta'']{} & H(Y, G) \end{array}$$

is commutative, where $\mu_Z: Z \to H(Z, G)$ is the mapping of Theorem 9.

THEOREM 11. *Suppose that X and Y are k-spaces. Every continuous homomorphism $h: C(Y, G) \to C(X, G)$ which maps every constant function on Y into the corresponding constant function on X, induces a unique continuous mapping j of X into Y such that $j' = h$. Furthermore, if h is a topological isomorphism, then the induced mapping j is a homeomorphism.*

PROOF. Let h' be the mapping of $H(X, G)$ into $H(Y, G)$ defined by $h'(h_x) = h_x \circ h$ for each $h_x \in H(X, G)$. Since X and Y are k-spaces, μ_X and μ_Y are homeomorphisms by Theorem 9. If we define $j: X \to Y$ by setting $j = \mu_Y^{-1} \circ h' \circ \mu_X$, then the above diagram shows that j is continuous. Note that $j(x) = y$ if and only if $h(g)(x) = g(y)$ for each $g \in C(Y, G)$. If $j': C(Y, G) \to C(X, G)$ is the mapping defined by $j'(g) = g \circ j$ for each $g \in C(Y, G)$, it is easy to verify that $j' = h$.

If $r: X \to Y$ is any continuous mapping such that $r(x) \neq j(x)$ for some $x \in X$, then there exists an $f \in C(X, G)$ such that $f(r(x)) \neq f(j(x))$. Hence $r' \neq j'$, and the uniqueness of j follows.

Now if h is a topological isomorphism, then j is onto and one-to-one (cf. [**5**, 10.2]), and j^{-1} is continuous. Hence j is a homeomorphism of X onto Y, and the proof is completed.

As one may notice from the above proof, the introduction of the mapping j depends solely on the homeomorphism of the maps μ_X and μ_Y, and, as noted in Remark 10, the mapping μ is always a homeomorphism if $H(X, G)$ is endowed with the point-open topology which indeed coincides with the compact-open topology if the domain space is discrete [**3**]. With this remark, we can now prove Theorem 8 very easily; take discrete topologies for the groups $C(Y, G)$ and $C(X, G)$ then apply the proof of Theorem 11.

REMARK 12. In fact, if we define an S-pair (X, G) in a weaker form, (that is if we define (X, G) to be an S-pair if, for each closed subset C of X and $x \notin C$ there exists an f in $C(X, G)$ such that $f(x) \notin \mathrm{cl}[f(C)]$), then most of the results stated above, except perhaps Theorems 5 and 7, hold.

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