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WALLMAN-TYPE COMPACTIFICATIONS ON 0-DIMENSIONAL SPACES

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ABSTRACT. Let E be Hausdorff 0-dimensional, $\mathscr{D}$ the discrete space $\{0, 1\}$, and $\mathscr{N}$ the discrete space of all nonnegative integers. Every E-completely regular space X has a clopen normal base $\mathscr{F}$ with $X\backslash F \in \mathscr{F}$ for each $F \in \mathscr{F}$. The Wallman compactification $\omega(\mathscr{F})$ is $\mathscr{D}$-compact. Moreover, if an E-completely regular space X has a countably productive clopen normal base $\mathscr{F}$ with $X\backslash F \in \mathscr{F}$ for each $F \in \mathscr{F}$, then the Wallman space $\eta(\mathscr{F})$ is $\mathscr{N}$-compact. Hence, if X has such an $\mathscr{F}$, and is an $\mathscr{F}$-realcompact space, then X is $\mathscr{N}$-compact.

Recently, the relations between Stone-Čech compactifications and Wallman compactifications, those between realcompactifications and Wallman compactifications and those between E-compactifications and Wallman compactifications have been studied by Frink [**6**], Njastad [**9**], the Steiners [**11**], [**12**], Alo and Shapiro [**1**], [**2**], [**3**], [**4**], Piacun and Su [**10**], and some others.

A topological space is said to be 0-dimensional if it has a base consisting of clopen (both closed and open) subsets of X. For other notations and terminology see one of [**1**], [**2**], [**3**], [**4**], [**10**], [**11**] and [**12**], and [**8**].

Let $\mathscr{H}$ be a base for closed subsets of E. Let X be a T_0-space. Let $\mathscr{E}(\mathscr{H})$ be the family of all subsets of X of the form $f^{-1}[B]$ where for some positive integer n, $f \in C(X, E^n)$ and $B \in \mathscr{H}$. According to the definition of E-complete regularity (see [**8**]), X is E-completely regular iff $\mathscr{E}(\mathscr{H})$ is a base for the closed subsets of X.

From now on we will let E be a T_2 0-dimensional space with card $E \geqq 2$. According to [**8**], the following theorems are true:

THEOREM (MRÓWKA). *The following three statements are equivalent*:
(1) *X is a* 0-*dimensional T_0-space.*
(2) *X is E-completely regular.*
(3) *X is $\mathscr{D}$-completely regular.*

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THEOREM (MRÓWKA). *Let X be a compact, 0-dimensional T_0-space. Then X is $\mathscr{D}$-compact and X is E-compact.*

If $\mathscr{F}$ is a family of clopen subsets of X such that $X\backslash F \in \mathscr{F}$ for each $F \in \mathscr{F}$, then $\mathscr{F}$ is a base for the open sets of X iff $\mathscr{F}$ is a base for the closed sets of X. In the sequel we shall mainly be concerned with bases which are rings (closed under the operations of taking finite unions and finite intersections). We therefore make the following definition: $\mathscr{F}$ is called a *complemental base* on X iff all of the following are satisfied

(1) $\mathscr{F}$ is a family of clopen subsets of X.
(2) $X\backslash F \in \mathscr{F}$ for each $F \in \mathscr{F}$.
(3) $\mathscr{F}$ is a ring.
(4) $\mathscr{F}$ is a base for closed sets of X.

It is obvious that any complemental base is a normal base. Also, if X is E-completely regular and if $\mathscr{F}$ is a complemental base on E, then $\mathscr{E}(\mathscr{F})$ is a complemental base on X. Since the family of all clopen subsets of E is a complemental base, it follows that every E-completely regular space X has at least one complemental base. Conversely, if a T_0-space X has a complemental base, then X is necessarily 0-dimensional and so, by Mrówka's Theorem quoted above, X is E-completely regular.

In order to fix the notation, we repeat the construction of the Wallman spaces $\omega(\mathscr{F})$ and $\eta(\mathscr{F})$ which arise from a normal base $\mathscr{F}$. Thus, let $\mathscr{F}$ be a normal base on X. Let $\omega(\mathscr{F})$ be the set of all $\mathscr{F}$-ultrafilters and $\eta(\mathscr{F})$ the set of all $\mathscr{F}$-ultrafilters with the c.i.p. (countable intersection property). We now topologize $\omega(\mathscr{F})$ and $\eta(\mathscr{F})$ as follows: In $\omega(\mathscr{F})$, for each $F \in \mathscr{F}$ we define the set $F^* = \{\mathscr{O} \in \omega(\mathscr{F}) : F \in \mathscr{O}\}$. Then the set $\{F^* : F \in \mathscr{F}\}$ can be taken as a base for closed subsets of $\omega(\mathscr{F})$. Similarly, in $\eta(\mathscr{F})$, we define $F^{**} = \{\mathscr{O}^* \in \eta(\mathscr{F}) : F \in \mathscr{O}^*\}$ for each $F \in \mathscr{F}$. $\omega(\mathscr{F})$ and $\eta(\mathscr{F})$ with the described topologies are called Wallman spaces. In fact $\omega(\mathscr{F})$ is a Hausdorff compactification of X. (See **[6]**, **[11]**.) Let ϕ be the natural embedding of X into $\omega(\mathscr{F})$ (or $\eta(\mathscr{F})$) defined by identifying $\phi(x)$ with the $\mathscr{F}$-ultrafilter consisting of all $F \in \mathscr{F}$ that contain x, denoted by $\mathscr{O}_x = \{F \in \mathscr{F} : x \in F\}$. It is clear $\mathscr{O}_x \in \omega(\mathscr{F})$ and $\eta(\mathscr{F})$.

The following Lemmas A, B and C are easy to prove.

LEMMA A. *Let $\mathscr{F}$ be a normal base on X. Then:*
(a) $(F_1 \cap F_2)^* = F_1^* \cap F_2^*$ $((F_1 \cap F_2)^{**} = F_1^{**} \cap F_2^{**})$ *for all* $F_1, F_2 \in \mathscr{F}$.
(b) $\phi(F) = \phi(X) \cap F^*$ $(\phi(X) \cap F^{**})$ *for all F in* $\mathscr{F}$.
(c) $\mathrm{cl}_{\omega(\mathscr{F})} \phi(F) = F^*$ $(\mathrm{cl}_{\eta(\mathscr{F})} \phi(F) = F^{**})$.

The proof is similar to that of Lemma I in **[1]**.

Since $\mathscr{F}$ is a disjunctive family (see **[1]**) and X is T_1, the mapping ϕ is a one-one mapping of X onto the subspace $\phi(X)$ of $\omega(\mathscr{F})$ $(\eta(\mathscr{F}))$.

According to Lemma A(b), ϕ is both continuous and closed. Hence ϕ is a homeomorphism.

THEOREM A. *Let X be E-completely regular and let $\mathscr{F}$ be a complemental base on X. Then $\mathscr{F}^*=\{F^*:F\in\mathscr{F}\}$ ($\mathscr{F}^{**}=\{F^{**}:F\in\mathscr{F}\}$) is a complemental base on $\omega(\mathscr{F})$ ($\eta(\mathscr{F})$ resp.). Hence $\omega(\mathscr{F})$ ($\eta(\mathscr{F})$) is E-completely regular.*

PROOF. Observe that for each $F\in\mathscr{F}$, $\mathscr{O}\in\omega(\mathscr{F})\backslash F^*$ iff $F\notin\mathscr{O}$ iff $(X\backslash F)\in\mathscr{O}$ iff $\mathscr{O}\in(X\backslash F)^*$. Therefore, for each $F\in\mathscr{F}$, $\omega(\mathscr{F})\backslash F^*=(X\backslash F)^*$. It follows that $\mathscr{F}^*$ is a complemental base on $\omega(\mathscr{F})$. Similarly, $\mathscr{F}^{**}$ is a complemental base on $\eta(\mathscr{F})$. Since $\omega(\mathscr{F})$ and $\eta(\mathscr{F})$ are Hausdorff spaces, the theorem is true.

LEMMA B. *In addition to the conditions in Lemma* A, *if $\mathscr{F}$ is countably productive, then*:

(a) *For $F_n\in\mathscr{F}$, $n=1,2,\cdots$, $(\bigcup_{n=1}^{\infty}F_n)^{**}=\bigcup_{n=1}^{\infty}F^{**}$ and*

$$\left(\bigcap_{n=1}^{\infty}F_n\right)^{**}=\bigcap_{n=1}^{\infty}F_n^{**}.$$

(b) *If $\mathscr{A}$ is a $\mathscr{F}^*$- ($\mathscr{F}^{**}$-) ultrafilter* (*with the c.i.p.*) *then $\mathscr{O}=\{F:F^*\in\mathscr{A}\}$ ($\mathscr{O}^*=\{F:F^{**}\in\mathscr{O}\}$) is an $\mathscr{F}$-ultrafilter* (*with the c.i.p.*). *And conversely.*

The proof is similar to that of Theorem 1 in **[4]**.

THEOREM B. *Let X be an E-completely regular space and let $\mathscr{F}$ be a complemental base on X. Then the Wallman compactification $\omega(\mathscr{F})$ is E-compact and also is $\mathscr{D}$-compact.*

PROOF. Theorem A implies that $\omega(\mathscr{F})$ is E-completely regular. Now apply Mrówka's theorems quoted above using the known fact that $\omega(\mathscr{F})$ is a compact Hausdorff space.

In general, we do not know that if the Wallman spaces $\omega(\mathscr{F}')$, $\eta(\mathscr{F}')$ of an E-completely regular space generated by the ring $\mathscr{F}'$ of all E-closed subsets of X is E-completely regular. However according to Theorem A, we have

COROLLARY A. *If X is an E-completely regular space and if $\mathscr{F}$ is a complemental base on X, then the Wallman spaces $\omega(\mathscr{F})$ and $\eta(\mathscr{F})$ are E-completely regular.*

If, in particular, E is either $\mathscr{N}$, the discrete space of the nonnegative integers, or $\mathscr{D}$, the discrete space $\{0, 1\}$. An E-closed subset of X is a subset A of X such that there is a positive integer n and a continuous function $f\in C(X, E^n)$ such that $A=f^{-1}[F]$ for some closed subset F

of E^n. Since F is clopen in E^n, each E-closed subset A of X, and $X\backslash A$ are indeed E-clopen sets. Let $\mathscr{F}_1$ denote the family of all such E-clopen sets. Then, by [**8**, (3.18)] $\mathscr{F}_1$ is a ring. It is easy to show that $\mathscr{F}_1$ is a complemental base on X iff X is E-completely regular.

COROLLARY B. *Let E be $\mathscr{N}$ or $\mathscr{D}$. For any E-completely regular space X, the Wallman spaces $\omega(\mathscr{F}_1)$, $\eta(\mathscr{F}_1)$ arising out of the ring $\mathscr{F}_1$ of E-closed (indeed it is E-clopen) subsets of X is E-completely regular.*

In a recent paper Chew [**5**] has given the following characterization of $\mathscr{N}$-compactness. We recall it here.

THEOREM C. *In a 0-dimensional space X, the following are equivalent:*
(i) *X is $\mathscr{N}$-compact.*
(ii) *Every clopen ultrafilter on X with the c.i.p. is fixed, (i.e., has non-empty intersection).*
(iii) *The collection of all the countable clopen coverings of X is complete.*

According to Frolik [7], let $\alpha=\{\mathscr{U}\}$ be a collection of coverings of a space X. An α-Cauchy family $\mathscr{G}$ is a filter of subsets of X such that for every $\mathscr{U}\in\alpha$, there exist U in $\mathscr{U}$ and G in $\mathscr{G}$ with $U\supset G$. The collection α is complete iff $\mathscr{G}$ is fixed (i.e., $\bigcap\mathscr{G}\neq\varnothing$) for each α-Cauchy family $\mathscr{G}$.

The following lemmas are needed to show that there is a Wallman space $\eta(\mathscr{F})$ that is $\mathscr{N}$-compact.

LEMMA C. *Let X be E-completely regular and let $\mathscr{F}$ be a complemental base on X which is countably productive. Then every $\mathscr{F}^{**}$-ultrafilter with the c.i.p. is fixed.*

Proof is straightforward from Lemma B(b).

LEMMA D. *Let $\mathscr{B}$ be a base consisting of clopen subsets of X. If the collection β of all countable coverings from $\mathscr{B}$ is complete, then the collection α of all countable clopen coverings is complete.*

PROOF. Let $\mathscr{A}$ be an α-Cauchy family, and $\mathscr{V}\in\beta$ be arbitrary. Since $\beta\subset\alpha$, $\mathscr{V}\in\alpha$, and $\mathscr{A}$ is an α-Cauchy family, there are $V\in\mathscr{V}$, and $A\in\mathscr{A}$ such that $V\supset A$. Hence $\mathscr{A}$ is a β-Cauchy family so that $\bigcap\mathscr{A}\neq\varnothing$.

LEMMA E. *Let X be an E-completely regular space and let $\mathscr{B}$ be a complemental base on X. Then the collection β of all countable clopen coverings of X from $\mathscr{B}$ is complete iff every $\mathscr{B}$-ultrafilter with the c.i.p. is fixed.*

PROOF. *Necessity.* Let $\mathscr{A}$ be an ultrafilter of $\mathscr{B}$ with the c.i.p. Suppose that $\mathscr{A}$ is not a β-Cauchy family. Then there would be a $\mathscr{V}\in\beta$ such that each $V_i\in\mathscr{V}$ does not meet some member of $\mathscr{A}$, namely, A_i. (For since

$\mathscr{A}$ is an ultrafilter, if V_i meets each member of $\mathscr{A}$, then $V_i \in \mathscr{A}$. Thus, $\mathscr{A}$ would be a β-Cauchy family.) Hence $V_i \subset X \backslash A_i$ for each $i=1, 2, \cdots$. Then $X=\bigcup_{i=1}^{\infty} V_i=\bigcap \mathscr{V} \subset \bigcup_{i=1}^{\infty} (X\backslash A_i)=X\backslash\bigcap_{i=1}^{\infty} A_i$. This implies $\bigcap_{i=1}^{\infty} A_i=\varnothing$. This contradicts the fact that $\mathscr{A}$ has the c.i.p. Therefore, $\mathscr{A}$ is a β-Cauchy family, and $\bigcap \mathscr{A} \neq \varnothing$.

Sufficiency. Let $\mathscr{G}$ be a β-Cauchy family. Then there is a $\mathscr{B}$-ultrafilter $\mathscr{A}$ containing $\mathscr{G}$. Since $\mathscr{G}$ is a β-Cauchy family and since $\mathscr{G} \subset \mathscr{A}$, then $\mathscr{A}$ is a β-Cauchy family. Moreover, suppose that $A_1, A_2, \cdots$, are in $\mathscr{A}$ and have empty intersection. Then $\bigcup_{i=1}^{\infty} (X\backslash A_i)=X$, and $\mathscr{V}_0=\{X\backslash A_i: i=1, 2, \cdots\} \subset \mathscr{B}$ is in β. This would contradict the fact that $\mathscr{A}$ is a β-Cauchy family. Hence $\mathscr{A}$ is an ultrafilter with the c.i.p. and $\bigcap \mathscr{A} \neq \varnothing$. Therefore $\bigcap \mathscr{G} \neq \varnothing$.

THEOREM D. *Let X be an E-completely regular space and let $\mathscr{F}$ be a complemental base on X which is countably productive. Then the Wallman space $\eta(\mathscr{F})$ is $\mathscr{N}$-compact.*

PROOF. By Theorem A, $\mathscr{F}^{**}$ is a complemental base on $\eta(\mathscr{F})$. Lemma C says that every $\mathscr{F}^{**}$-ultrafilter with c.i.p. is fixed. Combining this with Lemmas D and E and Theorem C, $\eta(\mathscr{F})$ is $\mathscr{N}$-compact.

Note that an E-completely regular space is a Tychonoff space. Combining Theorem 3 of [**4**], and Theorem D, we have

COROLLARY C. *Let X be an E-completely regular space, let $\mathscr{F}$ be a complemental base on X which is countably productive and suppose that X is $\mathscr{F}$-realcompact. Then $X=\eta(\mathscr{F})$ and so X is $\mathscr{N}$-compact.*

REMARKS. (1) Any discrete space has a complemental base which is countably productive.

(2) If $X=\mathscr{N}$, then the family $\mathscr{F}_1$ of all E-closed subsets which indeed is the family of all subsets is a complemental base which is countably productive. By [**4**, Theorem 3] $\eta(\mathscr{F})=\nu X=\mathscr{N}$. However $\omega(\mathscr{F})=\beta X$. Hence $\eta(\mathscr{F})$ is $\mathscr{N}$-compact but it is not $\mathscr{D}$-compact.

REFERENCES

1. R. A. Alo and H. L. Shapiro, *A note on compactifications and seminormal spaces*, J. Austral. Math. Soc. **8** (1968), 102–108. MR **37** #3527.

2. ———, *Normal bases and compactifications*. Math. Ann. **175** (1968), 337–340. MR **36** #3312.

3. ———, *Wallman compact realcompact spaces*, Contributions to Extension Theory of Topological Structures (Proc. Sympos., Berlin, 1967), Deutsch. Verlag Wissen., Berlin, 1969, pp. 9–14. MR **40** #872.

4. ———, *$\mathscr{Z}$-real compactifications and normal bases*, J. Austral. Math. Soc. **9** (1969), 489–495. MR **39** #3455.

5. K. P. Chew, *A characterization of $\mathscr{N}$-compact spaces*, Proc. Amer. Math. Soc. **26** (1970), 679–682. MR **42** #2436.

6. O. Frink, *Compactification and semi-normal spaces*, Amer. J. Math. **86** (1964), 602–607. MR **29** #4028.

7. Z. Frolík, *A generalization of realcompact spaces*, Czechoslovak Math. J. **13** (88) (1963), 127–138. MR **27** #5224.

8. S. Mrówka, *Further results on E-compact spaces*. I, Acta. Math. **120** (1968), 161–185. MR **37** #2165.

9. O. Njastad, *On Wallman-type compactifications*, Math. Z. **91** (1966), 267–276. MR **32** #6404.

10. N. Piacun and L. P. Su, *Wallman compactifications on E-completely regular spaces*, Pacific J. Math. **45** (1973), 321–326.

11. E. F. Steiner, *Wallman spaces and compactifications*, Fund. Math. **61** (1967/68), 295–304. MR **36** #5899.

12. A. K. Steiner and E. F. Steiner, *Nest generated intersection rings in Tychonoff spaces*, Trans. Amer. Math. Soc. **148** (1970), 589–601. MR **41** #7637.

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