INEQUALITIES FOR A PERTURBATION THEOREM OF PALEY AND WIENER

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ABSTRACT. A classical theorem of Paley and Wiener states that the set of functions $\{e^{i\lambda_n t}\}_{n=-\infty}^{\infty}$ forms a basis for $L^2(-\pi, \pi)$ whenever the following condition is satisfied:

$$(*) \qquad \|\sum c_n(e^{i\lambda_n t} - e^{int})\|^2 \leqq \theta^2 \sum |c_n|^2 \qquad (0 \leqq \theta < 1).$$

It is known that $(*)$ holds whenever λ_n is real and $|\lambda_n - n| \leqq L < \frac{1}{4}$ $(-\infty < n < \infty)$, and may fail to hold if $|\lambda_n - n| = \frac{1}{4}$.

In this note we show, more generally, that the condition $|\lambda_n - n| < \frac{1}{4}$ is also insufficient to ensure $(*)$.

1. **Introduction.** One of the fundamental results from the theory of nonharmonic Fourier series states that the functions $e^{i\lambda_n t}$ $(-\infty < n < \infty)$ form a basis for $L^2(-\pi, \pi)$ if they satisfy an inequality of the form

$$(1) \qquad \|\sum c_n(e^{i\lambda_n t} - e^{int})\|^2 \leqq \theta^2 \sum |c_n|^2$$

for some θ $(0 \leqq \theta < 1)$ and all finite sequences $\{c_n\}$ [7, p. 109]. It is well known [8, p. 210] that condition (1) holds whenever

$$|\lambda_n - n| \leqq L < (\log 2)/\pi \qquad (-\infty < n < \infty).$$

In another direction, Levinson showed [6, p. 48] that if

$$(2) \qquad |\lambda_n - n| \leqq L < \tfrac{1}{4},$$

then every function in $L^2(-\pi, \pi)$ has a nonharmonic series expansion $f \sim \sum c_n e^{i\lambda_n t}$ which is equiconvergent with its ordinary Fourier series over any interval $[-\pi+\varepsilon, \pi-\varepsilon]$ for any positive ε. For λ_n real, the question of whether (2) implies (1) was answered in the affirmative by M. I. Kadec [5].

The purpose of this note is to show that the condition $|\lambda_n - n| < \frac{1}{4}$ is *not* sufficient to imply (1). Our proof is based on the result [6, p. 67]

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that if $\{\mu_n\}$ is given by

$$
\begin{aligned}
\mu_n &= n - \tfrac{1}{4}, & n &> 0, \\
&= 0, & n &= 0, \\
&= n + \tfrac{1}{4}, & n &< 0,
\end{aligned}
$$

(3)

then $\{e^{i\mu_n t}\}_{n \neq 0}$ is closed in $L^2(-\pi, \pi)$, and therefore (1) cannot hold with $\lambda_n = \mu_n$. Using the same example, Ingham had already established in [4, p. 378] that the inequality

$$
A \sum |c_n|^2 \leq \| \sum c_n e^{i\mu_n t} \|^2
$$

cannot hold for any positive A, thereby also showing that (1) fails for this choice of λ_n. In this paper, the following theorem is established.

THEOREM. *If $\{\mu_n\}$ is given by (3) and if $\{\lambda_n\}$ is a sequence of complex numbers for which $|\lambda_n - \mu_n| \to 0$ $(n \to \pm \infty)$, then (1) cannot hold.*

COROLLARY. *The condition $|\lambda_n - n| < \tfrac{1}{4}$ is not sufficient to ensure (1).*

2. **Fourier frames.** Duffin and Schaeffer [3, p. 343] have termed a set of functions $\{e^{i\lambda_n t}\}$ a *frame* over the interval $(-\gamma, \gamma)$ if there exist positive constants A and B such that

(4) $$A \int_{-\gamma}^{\gamma} |g(t)|^2 \, dt \leq \frac{1}{2\pi} \sum_n \left| \int_{-\gamma}^{\gamma} g(t) e^{i\lambda_n t} \, dt \right|^2 \leq B \int_{-\gamma}^{\gamma} |g(t)|^2 \, dt$$

for every function $g(t)$ in $L^2(-\gamma, \gamma)$. It follows from a theorem of Paley and Wiener [7, p. 13] that an equivalent characterization is that the inequalities

(5) $$A \int_{-\infty}^{\infty} |f(x)|^2 \, dx \leq \sum_n |f(\lambda_n)|^2 \leq B \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

hold for every function f which is entire of exponential type γ and such that $f(x) \in L^2(-\infty, \infty)$. It is clear from (4) that a frame is a complete set of functions in $L^2(-\gamma, \gamma)$.

The following lemmas were established in [3, pp. 346, 360].

LEMMA 1. *The removal of a vector from a frame leaves either a frame or an incomplete set.*

LEMMA 2. *If $\{e^{i\lambda_n t}\}$ is a frame over $(-\gamma, \gamma)$, then there exists a $\delta > 0$ such that $\{e^{i\gamma_n t}\}$ is a frame over the same interval whenever $|\gamma_n - \lambda_n| \leq \delta$.*

3. **Proof of the Theorem.** Let us suppose to the contrary that (1) does hold for some θ, $0 \leq \theta < 1$. Then $\{e^{i\lambda_n t}\}$ is a frame over the interval $(-\pi, \pi)$ [1]. By Lemma 2, there is a $\delta > 0$ such that $\{e^{i\gamma_n t}\}$ is a frame over $(-\pi, \pi)$ whenever $|\gamma_n - \lambda_n| \leq \delta$. Since $|\lambda_n - \mu_n| \to 0$, it follows that, for

some sufficiently large N, the set

$$\{e^{i\lambda_n t}\}_{|n|\leq N} \cup \{e^{i\mu_n t}\}_{|n|>N}$$

is a frame over the same interval.

It is well known [2, p. 98] that if f is entire of exponential type γ and if $f(x) \in L^2(-\infty, \infty)$, then

$$|f(x + iy)|^2 \leq e^{2\gamma|y|} \int_{-\infty}^{\infty} |f(x)|^2 \, dx.$$

It follows from (5) that the set

$$\{e^{i\lambda_n t}\}_{n=-N}^{N} \cup \{e^{i\mu_n t}\}_{n=-\infty}^{\infty}$$

is also a frame over $(-\pi, \pi)$. Since $\{e^{i\mu_n t}\}_{n\neq 0}$ is closed in $L^2(-\pi, \pi)$, repeated application of Lemma 1 shows that the set $\{e^{i\mu_n t}\}_{n\neq 0}$ must be a frame over $(-\pi, \pi)$. Again invoking Lemma 2, we get $\varepsilon > 0$ such that the set $\{e^{\pm i\gamma_n t}\}_{n=1}^{\infty}$ is a frame over $(-\pi, \pi)$ whenever $|\gamma_n - \mu_n| \leq \varepsilon$. We complete the proof by showing that this leads to a contradiction.

Let us form the function $F(z) = \prod_{n=1}^{\infty} (1 - z^2/\gamma_n^2)$. Then F is entire of exponential type π [2, p. 186], and it was shown by Levinson [6, p. 49] that if $\{\gamma_n\}$ satisfies the inequality $|\gamma_n - n| \leq L < \frac{1}{4}$, then $F(x) \in L^2(-\infty, \infty)$. Therefore, under these conditions, $\{e^{i\gamma_n t}\}_{n\neq 0}$ is not closed in $L^2(-\pi, \pi)$ and therefore cannot possibly be a frame.

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