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IMMERSION IN THE METASTABLE RANGE AND 2-LOCALIZATION

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ABSTRACT. Our purpose is to study immersion properties in the metastable range using the techniques of localization of homotopy types. The main theorem states that immersion of a manifold M in euclidean space in the metastable range depends only upon the homotopy type M_2, the localization of M at the prime 2.

Introduction. It follows from [**2**] that in the metastable range immersion of a closed manifold depends only upon its homotopy type. One has

THEOREM 0.1. *Let M and N be homotopy equivalent closed differentiable manifolds of dimension n. Suppose M immerses in $\boldsymbol{R}^{n+k}$ for some $k \geqq [n/2]+2$. Then so does N.*

Namely if $f: M \to \boldsymbol{R}^{n+k}$ is an immersion with normal bundle ν, then one can extend f to an immersion of the total space $E(\nu)$ of ν, $\bar{f}: E(\nu) \to \boldsymbol{R}^{n+k}$. The zero section $s: M \to E(\nu)$ is an embedding. If $\theta: N \to M$ denotes a homotopy equivalence, then, by [**2**], $s\theta$ is homotopic to an embedding since $\dim E(\nu) - \dim N = k \geqq [n/2]+2$; denote such an embedding by $\bar{\theta}: N \to E(\nu)$. Then $\bar{f}\bar{\theta}$ immerses N into $\boldsymbol{R}^{n+k}$.

Our purpose is to study the immersion properties in the metastable range using the technique of localization of homotopy types ([**3**], [**8**]), and to prove a stronger form of Theorem 0.1 involving only the homotopy type of M at the prime 2, denoted by M_2. In order to be able to localize M at 2 we will suppose that its homotopy type is simple, meaning that M is connected and that $\pi_1 M$ operates trivially on $\pi_\# M$. We prove

THEOREM 0.2. *Let M and N be connected, simple, orientable and closed differentiable manifolds of dimension n whose 2-localizations M_2 and N_2 are homotopy equivalent. Suppose M immerses in $\boldsymbol{R}^{n+k}$ for some $k \geqq [n/2]+1$. Then N immerses in $\boldsymbol{R}^{n+2[k/2]+1}$.*

COROLLARY 0.3. *Let M and N be as in 0.2 and assume that N is a π-manifold. Then M immerses in $\boldsymbol{R}^{n+2[(n+2)/4]+1}$.*

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In particular this gives the following immersion theorem for generalized spherical space forms (compare Theorem C of [6] in the case of Lens spaces).

COROLLARY 0.4. *Let $M=\Sigma^{2n+1}/G$, where G is a finite abelian group of odd order operating fixed point free and smoothly on the homotopy sphere Σ^{2n+1}. Then M immerses in $\boldsymbol{R}^{2[3(n+1)/2]}$.*

1. **Localization.** We will use the notation and results of [3] and [8]. If X is a connected simple CW homotopy type then X_p denotes its p-localization (p a prime or 0); there are canonical maps $X \to X_p$ respectively $X_p \to X_0$. We will need the following basic result of [3].

THEOREM 1.1. *Let W be a connected finite CW complex and X a connected simple CW complex of finite type. Then the set of pointed homotopy classes $[W, X]$ is the pullback of the diagram of sets*

$$\{[W, X_p] \to [W, X_0] \mid p \in P\},$$

P denoting the set of primes.

We will use this theorem in a situation where X is simply connected. In this case we do not have to distinguish between free and pointed homotopy classes of maps.

2. **Some facts about SO and SF.** In this section we reformulate some known results about the special orthogonal groups and $SF(q)$, the monoid of degree one pointed maps of S^q. By suspension there is an inclusion $SF(q) \subset SF(q+1)$. Let $SF = \bigcup SF(q)$. The canonical maps $SO(q) \to SF(q)$ and $SO \to SF$ induce a map of pairs

$$\theta : (SO, SO(q)) \to (SF, SF(q)).$$

By [5, 3.2] one has

LEMMA 2.1. *The canonical map $\theta_{\#} : \pi_n(SO, SO(q)) \to \pi_n(SF, SF(q))$ is an isomorphism if $n \leqq 2q-2$.*

In accordance with §1 we write SO_p for the p-localization of SO. The reader should not confuse SO_p with $SO(p)$.

LEMMA 2.2. *Let q be an odd integer and p an odd prime. Then*
(i) *$\pi_k(SO_p, SO(q)_p)=0$ for $k \leqq 2q$;*
(ii) *$\pi_k(SF_p, SF(q)_p)=0$ for $k \leqq (p-1)(q+1)-2$.*

PROOF. The first result follows by the relative Hurewicz theorem from $H_k(SO_p, SO(q)_p : \boldsymbol{Z})=0$ for $k \leqq 2q$. The second result follows from the isomorphisms $\pi_k SF(q) \cong \pi_{k+q} S^q$ and $\pi_k SF \cong \pi_k^{st} S^0$ and the fact that

$(\pi_{k+q}S^q)_p \to (\pi_k^{st}S^0)_p$ is an isomorphism (p and q odd) if $k<(p-1)(q+1)-2$ and surjective for $k=(p-1)(q+1)-2$.

We will use the following lemma on liftings in fiber spaces which is proved in [**9**, 3.2].

LEMMA 2.3. *Consider the following diagram of connected* CW *complexes in which the columns are fibrations*:

$$\begin{tikzcd} F_1 \arrow[r, "f"] \arrow[d, "i_1"'] & F_2 \arrow[d, "i_2"] \\ E_1 \arrow[r, "e"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "b"] & B_2 \end{tikzcd}$$

Suppose $f_\#:\pi_k F_1 \to \pi_k F_2$ *is an isomorphism for* $k<n$ *and suppose given a* CW *complex* X *of dimension* n *together with maps* $g:X\to B_1$, $h:X\to E_2$ *such that* $bg=p_2h$. *Then there exists a map* $\tilde{g}:X\to E_1$ *with* $p_1\tilde{g}=g$.

3. **Geometric dimension.** Let X be a connected finite CW complex. If $\alpha\in[X, BSO]=\widetilde{KO}(X)$ is an oriented stable bundle over X, then we denote by $J\alpha$, α_p, respectively $J\alpha_p$, the canonical image of α in $[X, BSF]$, $[X, BSO_p]$, respectively $[X, BSF_p]$. As usual, we define the geometric dimension of $\alpha\in[X, BSO]$ by

$$\mathrm{gd}(\alpha) = \min\{j \mid \alpha\in \mathrm{im}([X, BSO(j)] \to [X, BSO])\}.$$

Since $\pi_i(SO, SO(j))=0$ for $i<j$, we see that $\mathrm{gd}(\alpha)\leqq \dim X$. We also define, for $\beta\in[X, BSF]$,

$$\mathrm{gd}(\beta) = \min\{j \mid \beta\in \mathrm{im}([X, BSF(j)] \to [X, BSF])\}.$$

Again $\pi_i(SF, SF(j)) = 0$ for $i<j$, so $\mathrm{gd}(\beta)\leqq\dim X$. Similarly we define, for $\gamma\in[X, BSO_p]$, respectively $\delta\in[X, BSG_p]$,

$$\mathrm{gd}(\gamma) = \min\{j \mid \gamma\in \mathrm{im}([X, BSO(j)_p] \to [X, BSO_p])\},$$
$$\mathrm{gd}(\delta) = \min\{j \mid \delta\in \mathrm{im}([X, BSF(j)_p] \to [X, BSF_p])\}.$$

In the following, X will always denote a finite connected CW complex.

LEMMA 3.1. *Let* $\alpha\in[X, BSO]$ *and* $\dim X=n$. *Then for* p *odd*, $\mathrm{gd}(\alpha_p)\leqq 2[n/4]+1$.

PROOF. By Lemma 2.2, $(SO_p, SO(2[n/4]+1)_p)$ is $4[n/4]+2$ connected. Since $4[n/4]+2\geqq n-1$ we see that $\tilde{H}^i(X; \pi_{i-1}(SO_p, SO(2[n/4]+1)_p))=0$ for all i and hence every map $\alpha_p:X\to BSO_p$ lifts to $BSO(2[n/4]+1)_p$. Hence the result.

Recall that n is any positive integer and p is any odd prime. We next define an integer $\varepsilon(n, p)$ by the equation

$$[n/(p-1)] + \varepsilon(n,p) = \min\{2j+1 \mid 2j+1 \geqq ((n+1)/(p-1)) - 1\}.$$

Notice that $\varepsilon(n,p)=0$, 1 for all such n and p.

LEMMA 3.2. *Let $\beta \in [X, BSF]$, $\dim X=n$, and p be any odd prime. Then* $\operatorname{gd}(\beta_p) \leqq [n/(p-1)]+\varepsilon(n,p)$.

PROOF. $(SF_p, SF([n/(p-1)]+\varepsilon(n,p))_p)$ is

$$(p-1)([n/(p-1)] + \varepsilon(n,p) + 1) - 2$$

connected by Lemma 2.2. Since

$$(p-1)([n/(p-1)] + \varepsilon(n,p) + 1) - 2 \geqq n-1$$

we see that

$$\tilde{H}^i(X; \pi_{i-1}(SF_p, SF([n/p] + \varepsilon(n,p))_p)) = 0 \quad \text{for all } i.$$

Hence the result.

LEMMA 3.3. *Let $\alpha \in [X, BSO]$, $\dim X=n$, and suppose there exists a $k>[n/2]$ such that $\operatorname{gd}(J\alpha)\leqq k$. Then $\operatorname{gd}(\alpha)\leqq k$.*

PROOF. Since $\operatorname{gd}(J\alpha)\leqq k$ it follows by Lemma 2.3 that we can lift $\alpha: X\to BSO$ into $BSO(k)$ provided that $can: \pi_i(SO, SO(k))\to\pi_i(SF, SF(k))$ is an isomorphism for $i<n$. But this is the case by Lemma 2.1, since $k>[n/2]$ and therefore $2k-2\geqq n-1$.

LEMMA 3.4. *Let $\beta\in[X, BSF]$. If $\max\{\operatorname{gd}(\beta_p) \mid p\in P\}\leqq 2k+1$, then $\operatorname{gd}(\beta)\leqq 2k+1$.*

PROOF. Notice that

$$BSF(2k+1) \simeq \prod_{p\in P} BSF(2k+1)_p$$

since $\pi_i BSF(2k+1)$ is finite for all i. Hence the result.

PROPOSITION 3.5. *Let $\alpha\in[X, BSO]$, $\dim X=n$, $k>[n/2]$ and $\operatorname{gd}(J\alpha_2)\leqq k$. Then $\operatorname{gd}(\alpha)\leqq 2[k/2]+1$.*

PROOF. By hypothesis $\operatorname{gd}(J\alpha_2)\leqq k\leqq 2[k/2]+1$ and, by Lemma 3.2, $\operatorname{gd}(J\alpha_p)\leqq [n/2]+\varepsilon(n,3)\leqq 2[k/2]+1$ for all odd primes p. Hence 3.4 implies that $\operatorname{gd}(J\alpha)\leqq 2[k/2]+1$. Since $2[k/2]+1\geqq k>[n/2]$ we conclude by 3.3 that $\operatorname{gd}(\alpha)\leqq 2[k/2]+1$.

4. **The proof of Theorem 0.2 and corollaries.** For a connected closed differentiable manifold M denote by $\tau(M)\in[M, BO]=\widetilde{KO}(M)$ the

stable tangent bundle and by $\nu(M)=-\tau(M)$ the stable normal bundle. The following proposition is well known in the corresponding "global" situation [**1**, 3.6].

PROPOSITION 4.1. *Let M and N be connected, simple and closed differentiable manifolds and $\varphi:N_2\rightarrow M_2$ a homotopy equivalence. Then*

$$\varphi * J\nu(M)_2 = J\nu(N)_2$$

in $[N_2, BF_2]\cong[N, BF_2]$.

PROOF. Suppose $\varphi * J\nu(M)_2=w\neq J\nu(N)_2$. Let $J\nu(N)\in[N, BF]$ be the stable normal fibration. Since $BF\cong\prod_{p\in P} BF_p$ we can define an element $\theta\in[N, BF]$ by giving $\theta_p\in J\nu(N)_p$ for p odd and $\theta_2=w$. Note that $\theta= J\nu(N)$. We want this to lead to a contradiction by showing that the Thom complex N^θ is S-reducible. For $\alpha\in[N, BF]$ we define $(N_p)^{\alpha_p}$ in the obvious way: represent α_p by the S_p^k-fibration $pr:E_p\rightarrow N_p$, $k\geqq\dim M=\dim N$; then $(N_p)^{\alpha_p}$ is the S-type of the mapping cone of pr. There is a canonical S-map $(N^\alpha)_p\rightarrow(N_p)^{\alpha_p}$ which is an S-equivalence since it induces an isomorphism in homology. We call N^α S-reducible at p if the (p-local) top cell of $(N_p)^{\alpha_p}$ splits off stably. Clearly N^α is S-reducible if and only if it is S-reducible at p for all primes p (apply Theorem 1.1). Since $\theta_p=J\nu(N)_p$ for p odd we see that N^θ is S-reducible at all odd primes. Further $(N_2)^{\theta_2}= (N_2)^\theta\cong(M_2)^{J\nu(M)_2}$ and hence N^θ is S-reducible at 2. We conclude that N^θ is S-reducible. But by a theorem of Spivak [7] this implies that $\theta=J\nu(N)$ contradicting our assumption.

We can now prove our theorem. Let M and N be as stated in Theorem 0.2. Denote by $\tilde{\tau}(M)$ the unique lift of $\tau(M):M\rightarrow BO\cong BSO\times \boldsymbol{RP}^\infty$ to BSO, the universal cover of BO, and let $\tilde{\nu}(M)=-\tilde{\tau}(M)\in\widetilde{KSO}(M)$ be the oriented stable normal bundle. Since M immerses in $\boldsymbol{R}^{n+k}$ we have $\mathrm{gd}(J\tilde{\nu}(M)_2)\leqq\mathrm{gd}(\tilde{\nu}(M))\leqq k$. Let $\varphi:N_2\rightarrow M_2$ be a homotopy equivalence. By Proposition 4.1, $\varphi * J\nu(M)_2=J\nu(N)_2$ and hence $k\varphi * J\tilde{\nu}(M)_2=kJ\tilde{\nu}(N)_2$ for $k:BSF_2\rightarrow BF_2$ the canonical map. Since k is a 2-fold covering up to homotopy, with covering transformations homotopic to the identity, every homotopy class into BF_2 which lifts to BSF_2 lifts in a unique way. Hence we have $\varphi * J\tilde{\nu}(M)_2=J\tilde{\nu}(N)_2$. We conclude that $\mathrm{gd}(J\tilde{\nu}(N)_2)\leqq \mathrm{gd}(J\tilde{\nu}(M)_2)\leqq k$, and, by Proposition 3.5, $\mathrm{gd}(\tilde{\nu}(N))\leqq 2[k/2]+1$. By the theorem of Hirsch [**4**] this implies that N immerses with codimension $2[k/2]+1$.

The Corollary 0.3 follows immediately by observing that N immerses with codimension 1, in case N is a π-manifold. We can therefore apply the theorem with $k=[n/2]+1$.

For Corollary 0.4 one uses that the covering projection $\Sigma^{2n+1}\rightarrow M$ induces an equivalence $\Sigma_2^{2n+1}\cong M_2$, since G is of odd order; notice that M

is simple since the operations of G are all homotopic to the identity. The result now follows from Corollary 0.3 by observing that Σ^{2n+1} is a π-manifold.

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