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SYZYGIES IN $[y^p z]$

D. G. MEAD AND M. E. NEWTON[1]

ABSTRACT. For every $[y^p z]$ we obtain an infinite sequence of syzygies as well as the coefficients of some of the terms in the derivatives of these syzygies.

In the investigation [2] of the ideals $[y^2 z]$ and $[y^3 z]$, it was necessary to use the syzygies $py_1 zA + yz_1 A - yzA_1 = 0$ and

$$p[yy_2 z_1 - (p+1)y_1^2 z_1 - yy_1 z_2]A + (2py_1 z_1 + yz_2)yA_1 - y^2 z_1 A_2 = 0.$$

In this paper we generalize these results, obtaining an infinite family of syzygies in $[y^p z]$, and develop some of the properties of their coefficients.

Notation. Let $A = y^p z$ and, as is customary, use subscripts to represent differentiation of y, z, and A. The following special notation for multinomial coefficients will be useful. Let

$$M(m, a, k) = m!/(1!)^{a_1}(2!)^{a_2} \cdots (m!)^{a_m} k!.$$

We also let

$$Y(a) = y^{a_0} y_1^{a_1} \cdots y_m^{a_m}.$$

For a function $F(a_0, a_1, a_2, \cdots)$ we say the sum $\sum F(a_0, a_1, \cdots)$ is (m, a) if the summation is over all sequences of nonnegative integers $(a_0, a_1, a_2, \cdots)$ satisfying $\sum_{i=0} a_i = m$. If F is a function $F(k, a_0, a_1, \cdots)$ we say the sum $\sum F(k, a_0, a_1, \cdots)$ is (m, a, k) if the summation is over all nonnegative integers k and sequences of nonnegative integers $(a_0, a_1, \cdots)$ satisfying $k + \sum ia_i = m$.

Since $z = A/y^p$, it is easy to see that $z_m = P(y, A)/y^{p+m}$ where $P(y, A)$ is a polynomial in the y_i and A_j. In fact we have

$$y^{p+m} z_m = \sum c(a_0, \cdots, a_m, k) Y(a) A_k \tag{1}$$

where

$$c(a_0, \cdots, a_m, k) = (-1)^{m-a_0} \binom{p + m - a_0 - 1}{p-1, a_1, \cdots, a_m} M(m, a, k)$$

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and the sum is (m, a) and (m, a, k). Using

$$y^{p+m+1}z_{m+1} = yP_1(y, A) - (p + m)y_1P(y, A), \tag{2}$$

one can construct a proof of (1) by induction on m. The relation (1) is true for $m=0$ and, assuming (1) true for m, we separate $P_1(y, A)$ in (2) into three types of terms: (i) $a_0(y_1/y)Y(a)A_k$, (ii) $\sum_{i=1} a_i(y_{i+1}/y_i)Y(a)A_k$, and (iii) $Y(a)A_{k+1}$. We collect all terms from the right side of (2) which involve $y^{b_0}\cdots y_m^{b_m}A_l$ where $\sum b_i=m+1$ and $l+\sum ib_i=m+1$, and factoring out $(-1)^{m+1-b_0}\binom{p+m-b_0}{p-1,b_1,\cdots,b_m}M(m+1, b, l)$, we find the following coefficients for the respective types of terms:

(i) $-(b_0b_1)/[(p+m-b_0)(m+1)]$,
(ii) $(m+1)^{-1}\sum_{i=1}(i+1)b_{i+1}=(m+1-b_1-l)/(m+1)$,
(iii) $l(m+1)^{-1}$,

and for the last term in (2), $[-(p+m)(-1)b_1(p+m-b_0)^{-1}(m+1)^{-1}]$. Since the sum of these coefficients is unity, this completes the proof of (1).

THE GENERAL SYZYGY THEOREM. *For every* $m>0$,

$$\sum(-1)^{m-a_0-1}\binom{p+m-a_0-1}{p-1, a_1, \cdots, a_m}M(m-1, a, k)Y(a)z_mA_k$$
$$+\sum(-1)^{m-a_0}\binom{p+m-a_0-1}{p-1, a_1, \cdots, a_m}M(m, a, k)Y(a)z_{m-1}A_k = 0,$$

where the first sum is $(m-1, a, k)$, *the second is* (m, a, k), *and both sums are* (m, a).

PROOF. Expanding $A_m=(y^pz)_m$ we find

$$y^mz_{m-1}A_m = y^{m+p}z_{m-1}z_m + \sum y^mz_{m-1}\, d(c_0, \cdots, c_m, j)Y(c)z_j \tag{3}$$

where the sum is (p, c) and (m, c, j), with $j<m$, and, by Leibniz' rule, $d=\binom{p}{c_0,c_1,\cdots,c_m}M(m, c, j)$. Using the identity (1) on $y^{p+m-1}z_{m-1}$, we see the first term on the right side of (3) equals the negative of the first sum in the statement of the theorem. The identity (1) can also be applied to every term of the summation in (3) since

$$p - c_0 = \sum_{i=1} c_i \leqq \sum ic_i = m - j,$$

or $m+c_0\geqq p+j$. By (1),

$$y^{p+j}z_j = \sum(-1)^{j-b_0}\binom{p+j-b_0-1}{p-1, b_1, \cdots, b_m}M(j, b, k)Y(b)A_k,$$

the sum being (j, b) and (j, b, k), and therefore the summation in (3) can be rewritten

$$\sum\sum(-1)^{j-b_0}\binom{p}{c_0,\cdots,c_m}M(m,c,j)$$
$$\times\binom{p+j-b_0-1}{p-1,b_1,\cdots,b_m}M(j,b,k)Y(a)z_{m-1}A_k$$
$$=\sum(-1)^{j-b_0}\binom{p}{c_0,\cdots,c_m}\binom{p+j-b_0-1}{p-1,b_1,\cdots,b_m}M(m,a,k)Y(a)z_{m-1}A_k,$$

where we have let $b_0=a_0+p+j-m-c_0$, and for $i>0$, $b_i=a_i-c_i$, and the latter sum is (m, a), (m, a, k), (p, c), and (m, c, j) with $j<m$. Hence the monomial $y^m z_{m-1}A_m$ minus the summation in (3) can be rewritten $y^m z_{m-1}A_m+\sum e(a_0,\cdots,a_m,k)M(m,a,k)Y(a)z_{m-1}A_k$ where the sum is (m, a) and (m, a, k), with $k<m$, and

$$e(a_0,\cdots,a_m,k)$$
$$=\sum(-1)^{m+c_0-a_0-p-1}\frac{p}{m+c_0-a_0}\binom{m+c_0-a_0}{c_0,\cdots,c_m,a_1-c_1,\cdots,a_m-c_m},$$

the sum being (p, c) with $\sum ic_i>0$ (i.e., $c_0<p$).

The proof of the theorem will be complete once we show that

$$e(a_0,\cdots,a_m,k)=(-1)^{m-a_0}\binom{p+m-a_0-1}{p-1,a_1,\cdots,a_m}.$$

Note that the coefficient of x^p in $(1+x)^{-(m-a_0)}(1+x)^{a_1}\cdots(1+x)^{a_m}$ is

$$\sum(-1)^{c_0}\binom{m-a_0+c_0-1}{c_0}\binom{a_1}{c_1}\binom{a_2}{c_2}\cdots\binom{a_m}{c_m}=0,$$

where the sum is (p, c). This implies that

$$\sum_{c_0<p}(-1)^{c_0}\binom{m-a_0+c_0-1}{c_0}\binom{a_1}{c_1}\cdots\binom{a_m}{c_m}$$
$$=(-1)^{p+1}\frac{m-a_0}{p}\binom{m-a_0+p-1}{p-1}.$$

Multiplying both sides by $(-1)^{m-p-a_0-1}(p/(m-a_0))\binom{m-a_0}{a_1,\cdots,a_m}$, we obtain the desired result.

Derivatives of syzygies. The investigation of the ideals $[y^2z]$ and $[y^3z]$ requires knowledge of the syzygies and of the coefficients of some terms in the derivatives of the syzygies. In the following theorem we obtain some of the coefficients in the derivatives of the new syzygies just derived, and this generalizes some of the results in **[2]**.

THEOREM 2. *The coefficient of $y_r^m z_s A_t$ in the Tth derivative of the mth syzygy is*

$$\frac{T!\,(t-pr-s)}{(r!)^m(s-m+1)!\,t!}\prod_{i=0}^{m-2}(t-pr-i),$$

where $T=mr+s+t-2m+1$.

The proof of the theorem will be given following two preliminary lemmas.

LEMMA 1.

$$\sum(-1)^{m-a_0}\binom{p+m-a_0-1}{p-1,a_1,\cdots,a_m}M(m-1,a,k)$$
$$\times\frac{(r!)^m t!}{(r!)^{a_0}((r-1)!)^{a_1}\cdots((r-m)!)^{a_m}(t-k)!}=\prod_{i=0}^{m-2}(t-pr-i),$$

where the sum is (m,a) and $(m-1,a,k)$.

PROOF. The sum on the left is $(m-1)!$ times

$$\sum(-1)^{m-a_0}\binom{p+m-a_0-1}{m-a_0}\binom{m-a_0}{a_1,\cdots,a_m}\binom{r}{1}^{a_1}\cdots\binom{r}{m}^{a_m}\binom{t}{k},$$

the sum being (m,a) and $(m-1,a,k)$ and we will show this sum equals $\binom{t-pr}{m-1}$.

Note that

$$(1+x)^{-pr}=(1+u)^{-p}=\sum_{l=0}(-1)^l\binom{p+l-1}{l}u^l$$

where $u=(1+x)^r-1$. For each term of degree $m-k-1$ in x from $u^l=((1+x)^r-1)\cdots((1+x)^r-1)$ and for $i=1,2,\cdots$, let a_i be the number of factors u from which a term of degree i in x was chosen. It is clear that $\sum ia_i=m-k-1$ and that the coefficient of x^{m-k-1} in u^l is

$$\sum\binom{l}{a_1,\cdots,a_m}\binom{r}{1}^{a_1}\cdots\binom{r}{m}^{a_m},$$

the sum being over a_i satisfying $\sum ia_i=m-k-1$ and $\sum_{i=1}a_i=l$. Therefore the coefficient of x^{m-1} in $(1+x)^{t-pr}$ is

$$\sum(-1)^{m-a_0}\binom{p+m-a_0-1}{m-a_0}\binom{m-a_0}{a_1,\cdots,a_m}\binom{r}{1}^{a_1}\cdots\binom{r}{m}^{a_m}\binom{t}{k}$$

and we are done.

LEMMA 2.

$$\sum (-1)^{m-a_0}\binom{p+m-a_0-1}{p-1, a_1, \cdots, a_m} M(m, a, k) \times \frac{(r!)^m t!}{(r!)^{a_0}((r-1)!)^{a_1}\cdots((r-m)!)^{a_m}(t-k)!} = \prod_{i=0}^{m-1}(t-pr-i),$$

where the sum is (m, a) *and* (m, a, k).

The proof is essentially the same as the one just given and involves examining the coefficient of x^m in $(1+x)^{t-pr}$.

PROOF OF THE THEOREM. Differentiating the mth syzygy T times, we find the coefficient of $y_r^m z_s A_t$ is

$$\begin{aligned}&\sum (-1)^{m-a_0-1}\binom{p+m-a_0-1}{p-1, a_1, \cdots, a_m} M(m-1, a, k)\\ &\qquad\times \frac{T!}{(r!)^{a_0}((r-1)!)^{a_1}\cdots((r-m)!)^{a_m}(s-m)!\,(t-k)!}\\ &\qquad+\sum (-1)^{m-a_0}\binom{p+m-a_0-1}{p-1, a_1, \cdots, a_m} M(m, a, k)\\ &\qquad\times \frac{T!}{(r!)^{a_0}((r-1)!)^{a_1}\cdots((r-m)!)^{a_m}(s-m+1)!\,(t-k)!},\end{aligned}$$

where both sums are (m, a), the first sum is also $(m-1, a, k)$ and the second is (m, a, k). These are

$$-T!/((r!)^m(s-m)!\,t!) \quad \text{and} \quad T!/((r!)^m(s-m+1)!\,t!)$$

times the respective expressions in Lemmas 1 and 2, and the result of Theorem 2 follows from those of the two lemmas.

REFERENCES

1. H. Levi, *On the structure of differential polynomials and on their theory of ideals*, Trans. Amer. Math. Soc. **51** (1942), 532–568. MR **3**, 264.

2. M. E. Newton, *The differential ideals* $[y^p z]$, Proc. Amer. Math. Soc. **30** (1971), 229–234. MR **44** #2733.

3. J. F. Ritt, *Differential algebra*, Amer. Math. Soc. Colloq. Publ., vol. 33, Amer. Math. Soc., Providence, R.I., 1950. MR **12**, 7.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS, CALIFORNIA 95616

DEPARTMENT OF MATHEMATICS, THIEL COLLEGE, GREENVILLE, PENNSYLVANIA 16125