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SOME PHENOMENA IN HOMOTOPICAL ALGEBRA

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ABSTRACT. In [6] D. G. Quillen developed homotopy theory in categories satisfying certain axioms. He showed that many results in classical homotopy theory (of topological spaces) go through in his axiomatic set-up. The duality observed by Eckmann-Hilton in classical homotopy theory is reflected in the axioms of a model category. In [7] we developed the theory of numerical invariants like the Lusternik-Schnirelmann category and cocategory etc. for such model categories and in [8] we dealt with applications of this theory to injective and projective homotopy theory of modules as developed by Hilton [2], [3, Chapter 13].

Contrary to the general expectations there are many aspects of classical homotopy theory which cannot be carried over to Quillen's axiomatic set-up. This paper deals with some of these phenomena.

Introduction. For any topological group G it is well known [1], [5], that there exists a principal fibre space $E_G \to^{p_G} B_G$ with group G and total space E_G contractible. This suggests the following question. Suppose M is a group object in a model category $\mathscr{C}$ in the sense of Quillen [6]. Does there exist a fibration $E \to^{p} B$ in $\mathscr{C}$ with the property that E is contractible (i.e. to say $\pi(RQ(E), RQ(E))=0$ following the notation of Quillen [6]) with fibre of p isomorphic to M? We will give examples to show that, in general, this is false. Also we will illustrate that, given a cogroup object H in a model category $\mathscr{C}$, there need not exist a cofibration $A \to^{q} E$ in $\mathscr{C}$ with E contractible and cofibre of q isomorphic to H.

Actually it will turn out that the two model categories $\mathscr{C}$ and $\mathscr{F}$ that we mention in this connection (§1) will have the following additional properties.

(i) All the objects are simultaneously group objects and cogroup objects.

(ii) For every object A both ΣA and ΩA are contractible.

It can easily be shown that in the category $\mathscr{T}$ of topological spaces if G is a group object with ΣG contractible then G itself is contractible.

In §2 we characterise all CW-complexes X with the property that ΣX is contractible. They turn out to be "Moore CW-complexes" $M(\pi, 1)$

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for groups π satisfying $H_1(\pi)=0=H_2(\pi)$. On the other hand, if X is a 0-connected CW-complex with ΩX contractible, then X itself is contractible.

1. **The model categories $\mathscr{C}$ and $\mathscr{F}$.** Let $\mathscr{C}$ denote the category of all modules over a Dedekind domain Λ. Defining cofibrations, weak equivalences and fibrations to be respectively monomorphisms, i-homotopy equivalences in the sense of Hilton [2] and maps satisfying the lifting property (L.P.) below, the author showed in [8] that $\mathscr{C}$ is a model category satisfying the axioms M_0 to M_5 of Quillen [6].

(L.P.) A map $p:E\to B$ in $\mathscr{C}$ satisfies (L.P.) if given any $f:J\to B$ with J injective there exists a lift $g:J\to E$ of f (i.e. $pg=f$).

Let $\mathscr{F}$ be the category of finitely generated modules over a Principal Ideal Domain (PID). Defining fibrations, weak equivalences and cofibrations to be respectively epimorphisms, p-homotopy equivalences in the sense of Hilton [2] and maps satisfying the extension property (E.P.) mentioned below it was shown in [8] that $\mathscr{F}$ is a model category in the sense of Quillen.

(E.P.) A map $q:A\to E$ is said to have the (E.P.) if given any finitely generated free Λ-module F and any map $\alpha:A\to F$ there exists a map $\beta:E\to F$ satisfying $\beta q=\alpha$.

It is clear that for any M in $\mathscr{C}$ (resp. $\mathscr{F}$) $M\times M\xrightarrow{\mu}M$, $M\xrightarrow{\sigma}M$ defined by $\mu(x,y)=x+y$, $\sigma(x)=-x$ make M into a group object in $\mathscr{C}$ (resp. $\mathscr{F}$) with μ as the multiplication, σ as the inversion and $0:M\to M$ as the unit. Similarly, $M\xrightarrow{\nu}M\oplus M$ given by $\nu(x)=(x, x)$ makes M into a cogroup object in $\mathscr{C}$ (resp. $\mathscr{F}$) with σ as the inversion and $M\xrightarrow{0}M$ as the co-unit. The following were proved in [8].

(1) In $\mathscr{C}$ as well as $\mathscr{F}$ all the objects are fibrant and cofibrant.

(2) An object M of $\mathscr{C}$ (resp. $\mathscr{F}$) is contractible if and only if M is injective (respectively free).

(3) For any M in $\mathscr{C}$ as well as $\mathscr{F}$ both ΣM and ΩM are contractible.

(4) For M in $\mathscr{C}$ or $\mathscr{F}$

(a) Ind Cat $M=0=$Cocat M if and only if M is contractible.

(b) Ind Cat $M=\infty=$Cocat M whenever M is not contractible.

PROPOSITION 1.1. *Let $M\in\mathscr{C}$ (resp. $\mathscr{F}$).*

(i) *If there exists a fibration $p:E\xrightarrow{p}B$ with E contractible and fibre of p isomorphic to M, then M itself is contractible.*

(ii) *If there exists a cofibration $q:A\to E$ with E contractible and cofibre of q isomorphic to M, then M itself is contractible.*

PROOF. If there exists a fibration $E\xrightarrow{p}B$ with E contractible and fibre of p isomorphic to M then, from the definition of Cocat M, we see that Cocat $M \leqq 1$. Then 4(b) implies M is contractible. This proves (i). The proof of (ii) is exactly dual and hence omitted.

2. **Contractibility of ΣX.** We now consider the category $\mathscr{T}_*$ of pointed topological spaces. Unless otherwise mentioned the homology groups we consider are the singular homology groups.

PROPOSITION 2.1. *Let X be a topological space which is of the homotopy type of ΩY for some Y. Then ΣX is contractible if and only if X itself is contractible.*

PROOF. When X is contractible clearly ΣX also is. Assume ΣX contractible. Let $f: X \to \Omega Y$ be a homotopy equivalence. Then

$$[X, X] \xrightarrow[\simeq]{f_*} [X, \Omega Y] \simeq [\Sigma X, Y] = 0$$

since ΣX is contractible. Thus $[X, X] = 0$, and X is contractible.

COROLLARY 2.2. *Let G be a topological group. Then ΣG is contractible if and only if G itself is.*

PROOF. It is known that G is of the homotopy type of ΩB_G where B_G is a classifying space for G.

REMARK 2.3. When G is a group object in $\mathscr{T}_*$ the above corollary asserts that ΣG is contractible if and only if G itself is. Consider the model categories $\mathscr{C}$ and $\mathscr{F}$ introduced in §1. All the objects in $\mathscr{C}$ (or $\mathscr{F}$) are group objects; for any object M both ΣM and ΩM are contractible. By taking the base ring Λ to be the ring Z of integers we see immediately that not all M in $\mathscr{C}$ (resp. $\mathscr{F}$) are contractible.

DEFINITION 2.4. Given any group π not necessarily abelian we call a space X a "Moore space" of type $(\pi, 1)$; if X is arcwise connected, $\pi_1(X) \simeq \pi$ and $H_j(X) = 0$ for $j \geqq 2$.

This definition differs from the one given in [9] in only one respect. We allow π to be nonabelian. We denote a Moore space of type $(\pi, 1)$ by $M(\pi, 1)$. Let $H_i(\pi)$ denote the ith homology group of the group π with coefficients in Z (with trivial π-operators). The following is proved in [9].

PROPOSITION 2.5. *A Moore space $M(\pi, 1)$ exists if and only if $H_2(\pi) = 0$.*

The proof given in [9] is valid even if π is not abelian. When $H_2(\pi) = 0$ the construction in [9] actually gives an $M(\pi, 1)$ CW-complex.

PROPOSITION 2.6. *Let X be a* CW-*complex. Then ΣX is contractible if and only if X is an $M(\pi, 1)$ complex with $H_1(\pi) = 0 = H_2(\pi)$.*

PROOF. Assume ΣX contractible. If α is the cardinality of the set of arc components of X then $H_1(\Sigma X)$ is free abelian of rank $\alpha - 1$. Since

$H_1(\Sigma X)=0$ we see that $\alpha=1$. Thus X is 0-connected. Let π denote $\pi_1(X)$. Then from $0=H_{j+1}(\Sigma X)\simeq H_j(X)$ for $j\geqq 1$ we see that $H_1(X)\simeq\pi/[\pi,\pi]\simeq H_1(\pi)=0$ and $H_j(X)=0$ for $j\geqq 2$. Hence, X is an $M(\pi, 1)$ complex with $H_1(\pi)=0$. From Proposition 2.5 we get $H_2(\pi)=0$.

Conversely, assume X is an $M(\pi, 1)$ CW-complex with $H_1(\pi)=0$. ΣX is simply connected (Van Kampen theorem). From $H_{j+1}(\Sigma X)=H_j(X)$ for $j\geqq 1$, $H_j(X)=0$ for $j\geqq 2$ and $H_1(X)\simeq\pi/[\pi,\pi]\simeq H_1(\pi)=0$ we immediately see that $H_i(\Sigma X)=0$ for all $i\geqq 1$. By J. H. C. Whitehead ΣX is contractible.

REMARK 2.7. Finitely presentable groups π satisfying $H_1(\pi)=0=H_2(\pi)$ are known to be *the groups* which occur as the fundamental groups of smooth homology n-spheres ($n\geqq 5$) **[4]**. There are many such nontrivial groups.

Thus there are noncontractible CW-complexes X with ΣX contractible.

3. Contractibility of ΩX.

LEMMA 3.1. *Suppose X is of the homotopy type of a* 0*-connected* CW-*complex. Then ΩX is contractible if and only if X is.*

This is an immediate consequence of the relation $\pi_i(\Omega X)\simeq\pi_{i+1}(X)$ for $i\geqq 0$ and J. H. C. Whitehead's theorem.

EXAMPLE 3.2. Let A_1, A_2, A_3, A_4 be the subsets of the plane $\boldsymbol{R}^2$ given by

$$A_1 = \{(x, \sin x^{-1}) \mid 0 < x \leqq \tfrac{1}{2}\pi^{-1}\}, \qquad A_2 = \{(\tfrac{1}{2}\pi^{-1}, y) \mid -2 \leqq y \leqq 0\},$$
$$A_3 = \{(x, -2) \mid 0 \leqq x \leqq \tfrac{1}{2}\pi^{-1}\}, \qquad A_4 = \{(0, y) \mid -2 \leqq y \leqq 1\}.$$

Let $X=A_1\cup A_2\cup A_3\cup A_4$. Let $x_0=(0, 1)$ be chosen as the base point in X. It is known that the space $\Omega(X, x_0)$ is contractible. However X is not contractible. In fact, the Čech homology $\check{H}_1(X)\simeq Z$; whereas the singular homology group $H_1(X)=0$. Hence, X is not even of the homotopy type of a CW-complex.

REMARK 3.3. Suppose X is a 0-connected noncontractible space with $\Omega(X)$ contractible. From Lemma 3.1 we immediately get that such an X will not be of the homotopy type of a CW-complex.

REMARK 3.4. Let X be the space $A_1\cup A_2\cup A_3\cup A_4$ given in Example 3.2. Using the fact that Čech *cohomology* theory satisfies the axioms of Eilenberg-Steenrod we get, in the usual way as a consequence of the exactness homotopy and excision axioms, $\check{H}^{i+1}(\Sigma X)\simeq\check{H}^i(X)$ for $i\geqq 1$. In particular, $\check{H}^2(\Sigma X)\simeq\check{H}^1(X)\simeq Z$. Hence, ΣX is not contractible. The same argument (repeated) yields that none of the spaces $\Sigma^l X$ ($l\geqq 1$) is contractible.

It might be interesting to find an example of a topological space X such that both ΣX and ΩX are contractible but X itself is not.

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