PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 43, Number 2, April 1974

INVARIANT SUBSPACES FOR PRODUCTS OF HERMITIAN OPERATORS

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ABSTRACT. It is shown that a nonscalar operator which is the product of a Hermitian operator and a positive operator has a nontrivial hyperinvariant subspace; this is a slight generalization of a result of Suzuki's.

Suzuki [3] has shown that an operator T which is the product of two positive operators has a nontrivial hyperinvariant subspace; (i.e., a closed linear manifold, different from $\{0\}$ and the entire space, which is left invariant by every operator which commutes with T). We present a simple proof of a slight generalization of Suzuki's result.

THEOREM. *If T is the product of a positive operator and a Hermitian operator, (and T is not a multiple of the identity), then T has a nontrivial hyperinvariant subspace.*

PROOF. Assume that $T=RK$ where R is positive and K is Hermitian; (if the product occurs in the other order simply apply the following to T^*). If R or K has a nontrivial nullspace then so does T or T^*, and the nullspace of an operator is hyperinvariant. Assume that R and K are injective; they then also have dense ranges since they are Hermitian. Now R has a unique positive square root $R^{1/2}$. It is trivial to verify the fact that

$$TR^{1/2} = R^{1/2}(R^{1/2}KR^{1/2})$$

and

$$(R^{1/2}K)T = (R^{1/2}KR^{1/2})(R^{1/2}K).$$

Since $R^{1/2}K$ and $R^{1/2}$ are injective and have dense ranges, the operator T is quasi-similar to the Hermitian operator $R^{1/2}KR^{1/2}$ and has a nontrivial hyperinvariant subspace by the well-known result of Sz. Nagy-Foiaş (cf. [2], [1, p. 103]).

It appears to be much more difficult to prove that the product of two Hermitian operators has an invariant subspace; if the above were extended

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Received by the editors November 10, 1972.

AMS (MOS) subject classifications (1970). Primary 47A15.

Key words and phrases. Operators on Hilbert space, hyperinvariant subspace, positive, Hermitian.

to the product of a positive operator and a unitary operator then, by the polar decomposition, the invariant subspace problem would be solved.

REFERENCES

1. P. A. Fillmore, *Notes on operator theory*, Van Nostrand Reinhold Math. Studies, no. 30, Van Nostrand-Reinhold, New York, 1970. MR **41** #2414.

2. B. Sz.-Nagy and C. Foiaş, *Analyse harmonique des opérateurs de l'espace de Hilbert*, Masson, Paris; Akad. Kiadó, Budapest, 1967; English rev. transl., North-Holland, Amsterdam; American Elsevier, New York; Akad. Kiadó, Budapest, 1970. MR **37** #778; **43** #947.

3. N. Suzuki, *Reduction theory of operators on Hilbert space*, Indiana Univ. Math. J. **20** (1971), 953–958.

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