A NOTE ON SUBADDITIVE FUNCTIONS

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ABSTRACT. A short proof is given for the following result:
Let ϕ be subadditive on $(0, b)$ and $\phi(t) < t$, $t \in (0, b)$. Then
 $\sup_{a \leq t \leq b} \phi(t)/t < 1$ for $0 < a < b$.

An assumption made on certain, positive, subadditive functions in our earlier work [1] has recently been shown to be superfluous by C. S. Wong [2]. We present here an alternative and simple proof of his result.

THEOREM. *Let ϕ be a subadditive function on $(0, b)$ such that $\phi(t) < t$ for all $t \in (0, b)$. Then $\sup_{a \leq t \leq b} \phi(t)/t < 1$, for $0 < a < b$.*

PROOF. Assume the contrary. From the compactness of $[a, b]$, there exists a sequence $\{t_n\} \subseteq [a, b]$ such that $\lim_{n \rightarrow \infty} t_n = \bar{t}$ and $\lim_{n \rightarrow \infty} \phi(t_n)/t_n = 1$. Define $\chi(t) = t - \phi(t)$ which is, by hypothesis, positive on $[a, b]$. From the subadditivity of ϕ , we have $\chi(t) - \chi(s) \geq \chi(t-s)$ for $t > s$. In particular, χ is increasing on $[a, b]$. Thus,

$$0 = \lim_{n \rightarrow \infty} \left(1 - \frac{\phi(t_n)}{t_n} \right) = \lim_{n \rightarrow \infty} \frac{\chi(t_n)}{t_n} = \frac{1}{\bar{t}} \left(\lim_{n \rightarrow \infty} \chi(t_n) \right) \geq \frac{\chi(a)}{\bar{t}} > 0,$$

which is the desired contradiction.

We remark that the above proof does not use the fact that ϕ is positive.

REFERENCES

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2. C. S. Wong, *Subadditive functions*, Pacific J. Math. **36** (1971), 549–551. MR **43** #4974.

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