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THE ANTIAUTOMORPHISM OF THE STEENROD ALGEBRA

DONALD M. DAVIS

ABSTRACT. Combinatorial techniques are used to obtain some formulas for the canonical antiautomorphism of the Steenrod algebra.

In this note we shall prove some nice formulas involving the canonical antiautomorphism χ of the mod p Steenrod algebra [**1**].

THEOREM 1. $\chi(\mathcal{P}^{p^{n-1}+\cdots+p+1})=(-1)^n\mathcal{P}^{p^{n-1}}\cdots\mathcal{P}^p\mathcal{P}^1$.

THEOREM 2. *For* $n\geqq k$, $\chi(Sq^{2^n-k})=Sq^{2^{n-1}}\cdots Sq^{2^{k-1}}(\chi(Sq^{2^{k-1}-k}))$.

$\chi(Sq^{2^{k-1}-k}) = Sq^{2^{k-2}}\chi(Sq^{2^{k-2}-k}) + Sq^{2^{k-2}-1}Sq^{2^{k-3}-1}\cdots Sq^3Sq^1$.

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Let $S(i)$ denote the sum of all Milnor basis elements [**1**] of the form $\mathcal{P}^R$ in dimension i. In [**1**, Corollary 6] is it shown that $\chi(\mathcal{P}^i)=(-1)^iS(2i(p-1))$. Thus Theorem 1 will follow by induction once we have shown that $\mathcal{P}^{p^{n-1}}\cdot S(2(p^{n-1}-1))=S(2(p^n-1))$. Indeed we shall prove

PROPOSITION.

$$\mathcal{P}^m\cdot S(l)=\sum_R\binom{\sum p^ir_i}{pm}\mathcal{P}^R,$$

where the sum is taken over all sequences $R=(r_1,\cdots)$ *having* $\sum 2(p^i-1)r_i=l+2m(p-1)$.

Then we need merely to note that if $\sum 2(p^i-1)r_i=2(p^n-1)$, then $\sum p^ir_i=p^n-1+\sum r_i$ and $1\leqq\sum r_i\leqq 2(p^n-1)/2(p-1)$, and hence $p^n\leqq\sum p^ir_i\leqq p+\cdots+p^n$, so that

$$\binom{\sum p^ir_i}{p^n}\equiv 1\ (\mathrm{mod}\ p).$$

PROOF OF PROPOSITION. The product contains a term

$$\prod\binom{r_i}{a_i}\mathcal{P}^{(r_1,\cdots)}$$

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for each Milnor matrix

$$\begin{array}{|cccc}\hline & r_1-a_1 & r_2-a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots\end{array}$$

such that $\sum p^{i-1}a_i=m$. In summing these, it is useful to note that if $s_1, \cdots$ is a finite sequence of positive integers and B ranges over all sequences $b_1, \cdots$ such that $0\leqq b_i\leqq s_i$ and $\sum b_i=k$, then

$$\sum_B \prod_i \binom{s_i}{b_i} = \binom{\sum s_i}{k}.$$

This is proved by comparing coefficients of x^k in $\Pi(1+x)^{s_i}=(1+x)^{\Sigma s_i}$.

Also we shall make use of the well-known facts

$$\binom{r}{a} \equiv \binom{p^i r}{p^i a} \bmod p, \quad \text{and} \quad \binom{p^i r}{b} \equiv 0 \bmod p$$

if b is not divisible by p^i. These are proved by comparing coefficients of $x^{p^i a}$ (respectively x^b) in $(1+x)^{p^i r}\equiv(1+x^{p^i})^r$.

Thus we have

$$\begin{aligned}\mathscr{P}^m \cdot S(l) &= \sum_R \sum_A \prod_i \binom{r_i}{a_i} \mathscr{P}^R \equiv \sum_R \sum_A \prod_i \binom{p^i r_i}{p^i a_i} \mathscr{P}^R \\ &\equiv \sum_R \sum_B \prod_i \binom{p^i r_i}{b_i} \mathscr{P}^R = \sum_R \binom{\sum p^i r_i}{pm} \mathscr{P}^R\end{aligned}$$

Here R ranges over sequences $(r_1, \cdots)$ having $\sum 2(p^i-1)r_i=l+2m(p-1)$, A ranges over sequences $(a_1, \cdots)$ having $\sum p^{i-1}a_i=m$, and B ranges over sequences $(b_1, \cdots)$ having $\sum b_i=pm$.

Theorem 2 follows by similar techniques using the following lemmas, which are easily proved by induction.

Lemma 1. *If $\sum (2^i-1)r_i=2^n-k$ with $n\geqq k$, then $\sum r_i\geqq k$. If $n=k-1$, the above is true except for the case when all $r_i=1$.*

Lemma 2. *$Sq^{2^k-1}Sq^{2^{k-1}-1}\cdots Sq^3Sq^1$ equals the Milnor basis element having 1 in the first k components.*

Reference

1. J. Milnor, *The Steenrod algebra and its dual*, Ann. of Math. (2) **67** (1958), 150–171. MR **20** #6092.

Department of Mathematics, Northwestern University, Evanston, Illinois 60201