PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 1, May 1974

HOMOTOPY GROUPS OF THE ISOTROPY GROUPS OF ANNULUS

JONG P. LEE

ABSTRACT. We compute the isotopy groups of various subspaces of the isotropy group at an interior point of an annulus. We also prove that if a and x are interior points of a disk D, then $\pi_0[H(D-a, x)]=Z_2$ and $\pi_n[H(D-a, x)]=0$ for $n\geqq 1$ where $H(D-a, x)$ is the isotropy group at x.

1. **Introduction.** Let X be a topological space, and let $H(X)$ denote the group of homeomorphisms of X onto itself topologized by the compact open topology. The isotropy group at $x\in X$ will be denoted by $H(X, x)=\{h\in H(X)|h(x)=x\}$. The arc-component of the identity $H_0(X)$ is a normal subgroup of $H(X)$ and $H(X)/H_0(X)=\pi_0[H(X)]$ is the group of the arc-components of $H(X)$, which is called the isotopy group of $H(X)$. The isotopy groups for the subspaces of $H(X)$ are similarly defined. In this note we compute the isotopy groups of various subspaces of the isotropy group at an interior point of an annulus, and $\pi_n[H(D-a, x)]$ for $n\geqq 0$ where D is a disk and $a, x\in \mathrm{Int}(D)$.

2. **Preliminaries.** We state some fundamental lemmas which will be needed in the sequel.

LEMMA 2.1. *The space of homeomorphisms of a closed n-cell onto itself which leave the boundary of the n-cell pointwise fixed is contractible* [**1**, *p*. 406].

LEMMA 2.2. *The space of homeomorphisms of an annulus onto itself, which leave one of the boundary curves pointwise fixed, is contractible* [**4**, *p*. 526].

LEMMA 2.3. *If A is an annulus, then* $\pi_0[H(A)]=Z_2\times Z_2$ [**6**, *p*. 924].

DEFINITION 2.4. An isotopy between imbeddings f_0 and f_1, defined on a space X into a space Y, is a continuous map $G:X\times I\to Y$ such that the function G_t defined by $G_t(x)=G(x, t)$ is a homeomorphism for each

Received by the editors June 28, 1972 and, in revised form, April 7, 1973.
AMS (*MOS*) *subject classifications* (1970). Primary 57E05; Secondary 55E05.
Key words and phrases. Homeomorphism, homotopy group, isotopy group, isotropy group, winding number, homeotopy exact sequence.

$t \in I=[0, 1]$, and for all $x \in X$, $G(x, 0)=f_0(x)$ and $G(x, 1)=f_1(x)$. If each G_t is also surjective and G_0 is the identity, then G is an ambient isotopy. An isotopy which moves no point on $\mathrm{Bd}(X)$ is called a B-isotopy. $h \approx g$ $[h \approx_B g]$ will denote that h is isotopic [B-isotopic] to g.

LEMMA 2.5. *Let f_0, f_1 be imbeddings of S^1 in* $\mathrm{Int}(X)$ *such that $f_0 \simeq f_1$: $S^1, * \to X, *$, where X is a 2-manifold. If $f_0(S^1)$ does not bound a disk or Möbius band in X, then there is an ambient isotopy between f_0 and f_1 keeping the base point fixed and which is fixed outside a compact subset of* $\mathrm{Int}(X)$ [**2**, *p.* 91].

Let $A=S^1 \times I$ and $H^2(A)=\{h \in H(A) | h=e$ on $\mathrm{Bd}(A)\}$. H. Gluck [**3**] defined the winding number for a homeomorphism $h \in H^2(A)$ as follows. Let η be the isomorphism of $\pi_1(S^1, 0)$ with Z which takes the class of the path $f(t)=t$ onto 1. Let α be any path in $S^1 \times I$ from (0, 0) to (0, 1) and $P_1: S^1 \times I \to S^1$ the natural projection. Then $P_1(\alpha)$ is a closed path in S^1 based at 0. Hence $[P_1(\alpha)]$ is an element of $\pi_1(S^1, 0)$ and $\eta([P_1(\alpha)])=\omega(\alpha)$ is an integer. The integer $\omega(h\alpha)-\omega(\alpha)$ is independent of the path α for any $h \in H^2(A)$.

DEFINITION 2.6. Let h be a homeomorphism in $H^2(A)$ and α a path in A from (0, 0) to (0, 1). Then the integer $W[h; A]=\omega(h\alpha)-\omega(\alpha)$ is called the winding number of h on A.

We note that W defines a homomorphism $W: H^2(A) \to Z$. But it is shown that the kernel of W is the arc-component of the identity $H^2_0(A)$ and thus W is in fact an isomorphism of $H^2(A)$ onto Z [**3**, p. 314].

The space $H(X)$ of a manifold X is a fiber bundle over $\mathrm{Int}(X)$ with fiber $H(X, x)$. The homotopy sequence of this bundle is exact and McCarty [**5**] obtained the following exact sequence which is called the homeotopy exact sequence of X,

$$\cdots \xrightarrow{P_*} \pi_{n+1}(X, x) \xrightarrow{d_*} \pi_n[H(X, x)] \xrightarrow{i_*} \pi_n[H(X)] \xrightarrow{P_*} \pi_n(X, x) \longrightarrow \cdots$$

$$\cdots \xrightarrow{P_*} \pi_1(X, x) \xrightarrow{d_*} \pi_0[H(X, x)] \xrightarrow{i_*} \pi_0[H(X)] \xrightarrow{P_*} 0,$$

where, if X is locally compact, locally connected and Hausdorff, $P_*(\pi_1[H(X)])$ is the center of $\pi_1(X, x)$ [**5**, p. 302].

3. **Isotopy groups.** In what follows A will denote an annulus which we take as the cylinder $S^1 \times I$ with the notations $C_1=S^1 \times 0$ and $C_2=S^1 \times 1$, $A_1=S^1 \times [0, \frac{1}{2}]$ and $A_2=S^1 \times [\frac{1}{2}, 1]$.

THEOREM 3.1. *Let a be an interior point of A. Then*

(i) $\pi_0[H(A, a)]=Z_2 \times Z_2$,

(ii) $\pi_0[H^1(A, a)]=Z$, *where* $H^1(A, a)=\{h \in H(A, a) | h=e \text{ on } C_1\}$,

(iii) $\pi_0[H^2(A, a)]=Z \times Z$, *where* $H^2(A, a)=\{h \in H(A, a) | h=e \text{ on } C_1 \cup C_2\}$.

PROOF. (i) In the homeotopy exact sequence of A, $P_*:\pi_1[H(A)]\to \pi_1(A, a)$ is onto. Thus the sequence yields $\pi_0[H(A, a)]=\pi_0[H(A)]$ and (i) holds by Lemma 2.3.

(ii) Let $a=(0, \frac{1}{2})$ and γ be the closed arc $S^1\times\frac{1}{2}$ in A. We note that every $h\in H^1(A, a)$ is orientation preserving on A since it is the identity on the boundary curve C_1. Thus $h(\gamma)$ does not bound a disk or Möbius band and $h(\gamma)\simeq\gamma$ fixing the point a, since the homotopy group $\pi_1(A, a)=Z$ has only the identity and inverse automorphisms and h must induce the identity automorphism of $\pi_1(A, a)$. Lemma 2.5 implies that there is an ambient isotopy $G_t:A, a\to A, a$ $(0\leqq t\leqq 1)$ such that $G_0=e$ and $G_1=h$ on γ. Since $G_1^{-1}h=e$ on γ, Lemma 2.2 implies that $G_1^{-1}h|A_2\approx e$ on A_2 by an isotopy which is the identity on the closed arc γ and moves only on the boundary curve C_2.

In A_1, since $G_1^{-1}h=e$ on $\gamma\cup C_1$, the isotopy classes of the restricted homeomorphisms $\{G_1^{-1}h|A_1\}$, for all $h\in H^1(A, a)$ and the above defined homeomorphisms G_1, are Z classified by the winding numbers $\{W[G_1^{-1}h|A_1; A_1]\}$. But a homeomorphism $G_1^{-1}h$ such that $W[G_1^{-1}h|A_1; A_1]\neq 0$ cannot be isotopic to the identity on $A=A_1\cup A_2$, since the isotopy in $H^1(A, a)$ leaves $C_1\cup\{a\}$ pointwise fixed. Thus the isotopy classes of the collection of all such homeomorphisms $G_1^{-1}h$ on A are Z. But since G_1 is B-isotopic to the identity fixing the point a, we have $\pi_0[H^1(A, a)]\cong \pi_0[\{G_1^{-1}h\}]$ and the isotopy group is Z.

(iii) We note that, for every $h\in H^2(A, a)$, there is a homeomorphism g such that $g\approx_B e$ on A and $g^{-1}h=e$ on $\gamma\cup C_1\cup C_2$ by Lemma 2.5. The collection of the restricted homeomorphisms $\{g^{-1}h|A_i\}$, for all $h\in H^2(A, a)$ and the homeomorphisms g, generates the isotopy classes Z for $i=1, 2$ by arguments similar to that of the proof of (ii). Since the homeomorphisms g are B-isotopic to the identity fixing the point a, we have $\pi_0[H^2(A, a)]\cong \pi_0[\{g^{-1}h\}]$ and thus the isotopy group is $Z\times Z$.

In the following theorem, the solution for the case $n=0$ partially answers a question raised by Quintas [**6**, p. 932].

THEOREM 3.2. *Let D be a disk and a, x be two different points in* $\mathrm{Int}(D)$. *Then*

$$\begin{aligned}\pi_n[H(D-a, x)] &= Z_2 \quad \text{if } n=0,\\ &= 0 \quad \text{if } n\geqq 1.\end{aligned}$$

PROOF. For the case $n=0$, for simplicity we consider $S^1\times(0, 1]$ for $D-a$ and let $x=(0, \frac{1}{2})$ and $C=S^1\times 1$. We first show that every $h\in H^+(D-a, x)$ is isotopic to the identity by an isotopy in $H^+(D-a, x)$, where $H^+(D-a, x)$ is the collection of the orientation preserving homeomorphisms in $H(D-a, x)$. Denote $\gamma=S^1\times\frac{1}{2}$, $B_1=S^1\times(0, \frac{1}{2}]$ and

$B_2=S^1\times[\frac{1}{2},1]$. Then it can be seen that $h(\gamma)$ does not bound a disk or Möbius band and $h(\gamma)\simeq\gamma$ fixing the point x. Thus by Lemma 2.5, there is an ambient isotopy $G_t\colon D-a, x\to D-a, x$ $(0\leqq t\leqq 1)$ such that $G_0=e$ and $G_1^{-1}h=e$ on the arc γ. Now observe that the restricted homeomorphism $G_1^{-1}h|B_1$ is isotopic to the identity on B_1. We can regard $G_1^{-1}h$ as a homeomorphism of a disk onto itself fixing one interior base point [**5**, Lemma 4.2]. Thus Lemma 2.1 implies that the homeomorphism $G_1^{-1}h$ is isotopic to the identity on the disk by an isotopy fixing the base point and the boundary γ pointwise. In B_2, by Lemma 2.2, it can be seen that the restricted homeomorphism $G_1^{-1}h|B_2$ is also isotopic to the identity on B_2 by an isotopy fixing the closed arc γ pointwise and moving on the boundary C. Thus $G_1^{-1}h$ is isotopic to the identity on $D-a$ by an isotopy fixing the point x, and thus the homeomorphism h is isotopic to the identity in $H^+(D-a, x)$. Hence we see that

$$\pi_0[H(D-a,x)] = H(D-a,x)/H^+(D-a,x)\cong Z_2$$

which completes the proof for the case $n=0$.

For the case $n\geqq 1$ we consider the homeotopy exact sequence. The sequence for $D-a$ yields $\pi_n[H(D-a)]=\pi_n[H(D-a,x)]$ for $n\geqq 2$ since $\pi_n(D-a,x)=0$ for $n\geqq 2$, and thus $\pi_n[H(D-a,x)]=0$ for $n\geqq 2$ [**6**, Theorem 5.1]. For $\pi_1[H(D-a,x)]$ we consider the end of the exact sequence,

$$\begin{aligned}\cdots\to\pi_2(D-a,x)\to\pi_1[H(D-a,x)]\to\pi_1[H(D-a)]\\ \to\pi_1(D-a,x)\to\pi_0[H(D-a,x)]\to\pi_0[H(D-a)]\to 0\end{aligned}$$

which is explicitly as follows.

$$\cdots\longrightarrow 0\xrightarrow{d_*}\pi_1[H(D-a,x)]\xrightarrow{i_*}Z\xrightarrow{P_*}Z\xrightarrow{d_*}Z_2\xrightarrow{i_*}Z_2\xrightarrow{P_*}0.$$

In this sequence, since $P_*[\pi_1(H(D-a))]$ is the center of $\pi_1(D-a,x)=Z$, P_* is an epimorphism which implies that it is in fact an isomorphism. Thus $\ker P_*=0$ and $i_*[\pi_1(H(D-a,x))]=0$. Hence $\ker i_*=\pi_1[H(D-a,x)]$, and since $d_*[\pi_2(D-a,x)]=0$, we obtain the result $\pi_1[H(D-a,x)]=0$. This completes the proof.

References

1. J. W. Alexander, *On the deformation of an n-cell*, Proc. Nat. Acad. Sci. U.S.A. **9** (1923), 406–407.

2. D. B. A. Epstein, *Curves on 2-manifolds and isotopies*, Acta Math. **115** (1966), 83–107. MR **35** #4938.

3. H. Gluck, *The embedding of two-spheres in the four-sphere*, Trans. Amer. Math. Soc. **104** (1962), 308–333. MR **26** #4327.

4. M. E. Hamstrom and E. Dyer, *Regular mappings and the space of homeomorphisms on a 2-manifold*, Duke Math. J. **25** (1958), 521–531. MR **20** #2695.

5. G. S. McCarty, Jr., *Homeotopy groups*, Trans. Amer. Math. Soc. **106** (1962), 293–304. MR **26** #3062.

6. L. V. Quintas, *Solved and unsolved problems in the computation of homeotopy groups of 2-manifolds*, Trans. New York Acad. Sci. **30** (1968), 919–938.

Department of Mathematics, Ohio State University, Lima, Ohio 45804

Current address: Department of Mathematics, State University of New York, Old Westbury, New York 11568