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ON (n, n)-ZEROS OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS OF ORDER $2n$

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ABSTRACT. Sufficient conditions on the coefficients p_{2n}, $p_{2n-1}, \cdots, p_0$ are given which guarantee that no nontrivial solution of $p_{2n}y^{(2n)}+p_{2n-1}y^{(2n-1)}+\cdots+p_0y=0$ has two distinct zeros each of order at least n. These conditions are in the form of n inequalities which are satisfied by linear combinations of the coefficients and their derivatives.

We study the real $(2n)$th order linear differential equation

$$p_{2n}(x)y^{(2n)} + p_{2n-1}(x)y^{(2n-1)} + \cdots + p_0(x)y = 0 \tag{1}$$

on an interval I where $p_{2n}(x)>0$ for all $x \in I$ and $p_i \in C^i(I)$ for $i=0, 1, \cdots, 2n$.

As usual, let $y^{(0)}(x)=y(x)$ and let $\binom{i}{k}=(i!)/[k!(i-k)!]$ for all nonnegative integers i and k with $i\geqq k$. A nontrivial solution $y(x)$ of (1) is said to have (n, n)-zeros in I if there exist two distinct points x_1 and x_2 in I with $y^{(j)}(x_i)=0$, $i=1, 2; j=0, 1, \cdots, n-1$. Let (Q_0) be the inequality

$$2(-1)^n p_0(x) + \sum_{k=1}^{2n} (-1)^{n+k} p_k^{(k)}(x) \geqq 0 \qquad (x \in I) \tag{Q_0}$$

and, for each i with $1\leqq i\leqq n-1$, let (Q_i) be the inequality

$$\sum_{k=0}^{2n-2i} (-1)^{n+k+i}\left[\binom{i+k}{k} + \binom{i+k-1}{k}\right] p_{2i+k}^{(k)}(x) \geqq 0 \qquad (x \in I). \tag{Q_i}$$

Theorem 1, the main result of this work, shows that no nontrivial solution of (1) has (n, n)-zeros in I provided that the coefficients in (1) satisfy the n inequalities (Q_i), $i=0, 1, \cdots, n-1$. The works of many authors relate in some way to the existence or nonexistence of solutions of (1) having (n, n)-zeros. Some specific references are Leighton and Nehari [5], Barrett [1], Reid [8], Hunt [4], Levin [6], Coppel [2], and Swanson

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[9]. An important aspect of the work of this paper is that the main theorem is proved without assuming equation (1) is selfadjoint while most of the work done to date requires selfadjointness.

Two corollaries are given, the first being an application of Theorem 1 to selfadjoint equations—an application which yields a well-known result. Corollary 2 is a factorization result. Aside from the classical paper of Pólya [7], some more recent work on factorization has been done by Barrett [1], Heinz [3], and Zettl [10], [11], [12].

The proof of Theorem 1 is an extension of the method employed by Leighton and Nehari [5] to show that no nontrivial solution of

$$(2)\quad (r(x)y'')''+p(x)y=0,\qquad r(x)>0, p(x)>0, r\in C^2(I), p\in C(I),$$

has two double zeros. They proved this result by displaying, for each solution $y(x)$ of (2), an auxiliary function $\phi_y(x)$ which is strictly increasing on I and is zero at each point of I where $y(x)$ has a double zero.

THEOREM 1. *If the coefficients* $p_{2n}, p_{2n-1}, \cdots, p_0$ *in* (1) *satisfy* (Q_i) *for* $0\leq i\leq n-1$, *then no nontrivial solution of* (1) *has* (n, n)*-zeros in* I.

PROOF. For each nontrivial solution $y(x)$ of (1), let $\phi_y(x)$ be the auxiliary function defined by

$$(3)\quad \begin{aligned}\phi_y(x) &= a_{0(2n-1)}(x)p_{2n}(x)y(x)y^{(2n-1)}(x)\\ &\quad+\sum_{i=0}^{2n-2}a_{0i}(x)y(x)y^{(i)}(x)+\sum_{j=1}^{n-1}\sum_{i=j}^{2n-1-j}a_{ji}(x)y^{(j)}(x)y^{(i)}(x)\end{aligned}$$

where the functions $a_{ji}, j=0,\cdots,n-1;\ i=j,\cdots,2n-1-j$, will be determined later. Note that, if $y(x)$ is a solution of (1) which has a zero of order at least n at a point $a\in I$, then $\phi_y(a)=0$.

We will now exhibit a choice of the a_{ji}'s which makes $\phi_y(x)$ nondecreasing as x increases in I. Supposing each a_{ji} is differentiable, we see that the derivative of ϕ_y is

$$(4)\quad \begin{aligned}\phi_y' &= (a_{0(2n-1)}p_{2n})'yy^{(2n-1)}+a_{0(2n-1)}p_{2n}y'y^{(2n-1)}\\ &\quad+\sum_{k=0}^{2n-1}(-a_{0(2n-1)}p_k yy^{(k)})+\sum_{i=0}^{2n-2}[a_{0i}'yy^{(i)}+a_{0i}y'y^{(i)}+a_{0i}yy^{(i+1)}]\\ &\quad+\sum_{j=1}^{n-1}\sum_{i=j}^{2n-1-j}[a_{ji}'y^{(j)}y^{(i)}+a_{ji}y^{(j+1)}y^{(i)}+a_{ji}y^{(j)}y^{(i+1)}].\end{aligned}$$

After collecting terms, we let A_{ji} $(j=0,\cdots,n;\ i=j,\cdots,2n-j)$ be the expression which is the coefficient of $y^{(j)}y^{(i)}$ in the right-hand side of (4) (take $A_{0(2n)}=0$ since $yy^{(2n)}$ does not appear in the right-hand side of (4)).

We now show it is possible to choose the a_{ji}'s so that $A_{ji}(x)=0$ when $i\neq j$ and $A_{ji}(x)\geqq 0$ when $i=j$ for all $x\in I$; therefore, with this choice of the a_{ji}'s, we will have $\phi'_y(x)\geqq 0$ for all $x\in I$. Since we want $A_{ji}=0$ for $i\neq j$, the a_{ji}'s must satisfy a system of simultaneous equations. Letting (E_{ji}) for $i\neq j$ represent the equation $A_{ji}=0$, we obtain the system

$$\begin{array}{lcccccccl}
(E_{01}) & 2a_{00} & + & a'_{01} & - & a_{0(2n-1)}p_1 & = 0 \\
(E_{02}) & a_{01} & + & a'_{02} & - & a_{0(2n-1)}p_2 & = 0 \\
\cdot & & & & & & \\
\cdot & & & & & & \\
\cdot & & & & & & \\
(E_{0(2n-1)}) & a_{0(2n-2)} & + & (a_{0(2n-1)}p_{2n})' & - & a_{0(2n-1)}p_{2n-1} & = 0 \\
(E_{12}) & a_{02} & + & 2a_{11} & + & a'_{12} & = 0 \\
(E_{13}) & a_{03} & + & a_{12} & + & a'_{13} & = 0 \\
\cdot & & & & & & \\
\cdot & & & & & & \\
\cdot & & & & & & \\
(E_{1(2n-2)}) & a_{0(2n-2)} & + & a_{1(2n-3)} & + & a'_{1(2n-2)} & = 0 \\
(E_{1(2n-1)}) & a_{0(2n-1)}p_{2n} & + & a_{1(2n-2)} & & & = 0 \\
(E_{23}) & a_{13} & + & 2a_{22} & + & a'_{23} & = 0 \\
(E_{24}) & a_{14} & + & a_{23} & + & a'_{24} & = 0 \\
\cdot & & & & & & \\
\cdot & & & & & & \\
\cdot & & & & & & \\
(E_{2(2n-3)}) & a_{1(2n-3)} & + & a_{2(2n-4)} & + & a'_{2(2n-3)} & = 0 \\
(E_{2(2n-2)}) & a_{1(2n-2)} & + & a_{2(2n-3)} & & & = 0 \\
\cdot & & & & & & \\
\cdot & & & & & & \\
\cdot & & & & & & \\
(E_{(n-2)(n-1)}) & a_{(n-3)(n-1)} & + & 2a_{(n-2)(n-2)} & + & a'_{(n-2)(n-1)} & = 0 \\
(E_{(n-2)n}) & a_{(n-3)n} & + & a_{(n-2)(n-1)} & + & a'_{(n-2)n} & = 0 \\
(E_{(n-2)(n+1)}) & a_{(n-3)(n+1)} & + & a_{(n-2)n} & + & a'_{(n-2)(n+1)} & = 0 \\
(E_{(n-2)(n+2)}) & a_{(n-3)(n+2)} & + & a_{(n-2)(n+1)} & & & = 0 \\
(E_{(n-1)n}) & a_{(n-2)n} & + & 2a_{(n-1)(n-1)} & + & a'_{(n-1)n} & = 0 \\
(E_{(n-1)(n+1)}) & a_{(n-2)(n+1)} & + & a_{(n-1)n} & & & = 0
\end{array}$$

We obtain a solution of this system by setting $a_{(n-1)n}=p_{2n}$. It then follows from equations $(E_{(n-j)(n+j)})$ $(j=1,\cdots,n-2)$ that $a_{i(2n-1-i)}=(-1)^{n-i+1}p_{2n}$ $(i=1,\cdots,n-1)$. Equation $(E_{1(2n-1)})$ then implies that $a_{0(2n-1)}=(-1)^{n+1}$. We now use consecutively equations $(E_{0(2n-1)})$, $(E_{0(2n-2)}),\cdots,(E_{01})$ to solve for $a_{0(2n-2)}, a_{0(2n-3)},\cdots,a_{00}$. Then we use equations $(E_{1(2n-2)}),(E_{1(2n-3)}),\cdots,(E_{12})$ to determine $a_{1(2n-3)}, a_{1(2n-4)},\cdots,$ a_{11}. Continuing in the same fashion, we are able to find all the a_{ji}'s.

Summarizing these calculations, we see that

$$a_{i(2n-1-i)} = (-1)^{n-i+1}p_{2n} \quad \text{for } i=1,\cdots,n-1, \tag{5}$$

$$a_{0(2n-1)} = (-1)^{n+1}, \tag{6}$$

$$a_{ji} = \sum_{k=0}^{2n-i-j-1}(-1)^{n+j+k+1}\binom{j+k}{k}p_{i+j+k+1}^{(k)} \quad \text{for } 0 \leqq j < i \text{ and } i+j<2n-1, \tag{7}$$

and finally that

$$a_{ii} = \frac{1}{2}\sum_{k=0}^{2n-2i-1}(-1)^{n+i+k+1}\binom{i+k}{k}p_{2i+k+1}^{(k)} \quad \text{for } 0 \leqq i \leqq n-1. \tag{8}$$

Let the a_{ji}'s be chosen as in (5), (6), (7), and (8). Then ϕ_y' as expressed in (4) reduces to

$$\begin{aligned}\phi_y' &= \sum_{j=0}^{n} A_{jj}[y^{(j)}]^2 \\ &= (a_{00}' - a_{0(2n-1)}p_0)y^2 + \sum_{k=1}^{n-1}(a_{(k-1)k} + a_{kk}')[y^{(k)}]^2 + a_{(n-1)n}[y^{(n)}]^2.\end{aligned} \tag{9}$$

However, it can be seen that, for each j with $0\leqq j\leqq n-1$, $2A_{jj}$ is the left-hand side of the inequality (Q_j). Therefore, we see from the hypothesis together with $A_{nn}(x)=p_{2n}(x)>0$ for all $x\in I$ that $\phi_y'(x)\geqq 0$ for $x\in I$.

Suppose now that there exists a nontrivial solution $y(x)$ of (1) which has (n,n)-zeros in I and let x_1 and x_2 be points in I with $x_1<x_2$ and $y^{(j)}(x_i)=0$ $(i=1,2;\ j=0,\cdots,n-1)$. Then $\phi_y'(x)=0$ for all $x\in[x_1,x_2]$ since $\phi_y(x_1)=\phi_y(x_2)=0$. However, this implies $y^{(n)}(x)=0$ for all $x\in[x_1,x_2]$ which is impossible and completes the proof.

As already mentioned, the following corollary is well known and the proof is usually by variational techniques. It is valid under the less restrictive hypothesis that $p_i\in C^i(I)$ rather than $p_i\in C^{2i}(I)$ (for a proof, see Theorem 18 on p. 77 of **[2]**).

COROLLARY 1. *Suppose* $p_i\in C^{2i}(I)$ $(i=0,\cdots,n)$ *and, for all* $x\in I$, $p_i(x)\geqq 0$ $(i=0,\cdots,n-1)$ *and* $p_n(x)>0$. *Then no nontrivial solution of* *the selfadjoint equation*

$$\sum_{k=0}^{n}(-1)^k[p_k(x)y^{(k)}]^{(k)} = 0 \tag{10}$$

has (n, n)-zeros in I.

PROOF. The proof follows by expanding the left-hand side of (10) and applying Theorem 1 directly to the resulting equation. The details are omitted.

We now prove the already-mentioned Corollary 2.

COROLLARY 2. *Suppose $I=[a, b)$ and $J=(a, b)$. If $p_0, \cdots, p_{2n}$ where $p_i \in C^i(I)$ satisfy $p_{2n}(x)>0$ $(x \in I)$ and inequalities (Q_j) $(j=0, \cdots, n-1)$, then the $(2n)$th order operator L defined for $y \in C^{2n}(I)$ by*

$$Ly = p_{2n}y^{(2n)} + p_{2n-1}y^{(2n-1)} + \cdots + p_0 y$$

has a factorization on J of the form $L=L_1L_2$ where L_1 and L_2 are both normal nth order linear differential operators on J.

PROOF. Let $y_j(x)$ for $1 \leqq j \leqq n$ be the n linearly independent solutions of $Ly=0$ which satisfy the initial conditions $y_j^{(i)}(a)=\delta_{(n+j-1)i}$ $(j=1, \cdots, n;\ i=0, \cdots, 2n-1)$ where $\delta_{mk}=1$ if $m=k$ and $\delta_{mk}=0$ if $m \neq k$. Let $W_n(x)$ be the Wronskian matrix (a_{ij}) where $a_{ij}=y_j^{(i-1)}(x)$ for $i=1, \cdots, n$; $j=1, \cdots, n$. If $W_n(b)=0$ for some $b \in J$, then a nontrivial solution $y(x)$ of (1) can be found which satisfies $y^{(k)}(a)=0=y^{(k)}(b)$ for $k=0, \cdots, n-1$ thus contradicting Theorem 1. Therefore, $W_n(x) \neq 0$ for all $x \in J$ and, by a theorem of Zettl **[10]**, L has a factorization such as we desire on the interval J.

We remark that the same proof shows Corollary 2 is valid with $I=[a, b]$ and $J=(a, b]$. Furthermore, the same type of proof shows that if no solution of an nth order linear differential equation has $(n-k, k)$-zeros in $[a, b)$ (or $[a, b]$) where $0<k<n$, then the associated operator L has a factorization $L=L_1L_2$ on (a, b) (or $(a, b]$) where L_1 is of order $n-k$ and L_2 is of order k.

We conclude by exhibiting an easy way to construct inequalities (Q_0), $(Q_1), \cdots, (Q_{n-1})$. Construct the rectangular array

$$\begin{array}{cccccccc}
2 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\
2 & 3 & 4 & 5 & 6 & \cdot & \cdot & \cdot \\
2 & 5 & 9 & 14 & 20 & \cdot & \cdot & \cdot \\
2 & 7 & 16 & 30 & 50 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot
\end{array}$$

where the element b_{ij} in the ith row and jth column is given by $b_{i1}=2$ $(i=1, 2, 3, \cdots)$, $b_{1j}=1$ $(j=2, 3, 4, \cdots)$, and $b_{ij}=b_{(i-1)j}+b_{i(j-1)}$ when $i>1$ and $j>1$. Note that this is a "Pascal's triangle" of sorts where an interior element is obtained by adding the element immediately to the left of the given element to the element immediately above the given element.

The absolute values of the coefficients in the inequality (Q_i) where $0\leqq i\leqq n-1$ can be seen to be the first $2n-2i+1$ numbers in the ith row of this array. The remaining information necessary to write (Q_j) is easily determined. For example, when $2n=6$, (Q_0), (Q_1), and (Q_2) reduce to

$$-2p_0 + p_1' - p_2'' + p_3''' - p_4^{(4)} + p_5^{(5)} - p_6^{(6)} \geqq 0,$$
$$2p_2 - 3p_3' + 4p_4'' - 5p_5''' + 6p_6^{(4)} \geqq 0, \quad \text{and}$$
$$-2p_4 + 5p_5' - 9p_6'' \geqq 0, \quad \text{respectively.}$$

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