PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 1, May 1974

CONVOLUTIONS OF CONTINUOUS MEASURES AND SUMS OF AN INDEPENDENT SET

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ABSTRACT. Let E be a compact independent subset of an l.c.a. group G; $\mu_1, \cdots, \mu_{n+1}$ continuous regular bounded Borel measures on G; and $k_1, \cdots, k_n$ integers. Let $k_i \times E = \{k_i x \mid x \in E\}$. We prove (1) $\mu_1 * \cdots * \mu_{n+1}(k_1 \times E + \cdots + k_n \times E) = 0$ (the proof is a combinatorial argument).

As a corollary of (1) we obtain (2) if H is any closed nondiscrete subgroup of G, then the intersection of H with the group generated by E has zero H-Haar measure.

1. In [**2**, Theorem 2], Hartman and Ryll-Nardzewski showed neatly that if μ_1 and μ_2 are continuous bounded measures on T, the circle group, and $E \subset T$ is a compact independent subset of T, then $\mu_1 * \mu_2(E) = 0$. In a more complicated manner, Salinger and Varopoulos proved this result for any metrizable group G in [**4**, Theorem 1]. We prove a generalization of this theorem which holds for an arbitrary nondiscrete l.c.a. group G.

We denote by Z the set of all integers, $Z^+ = \{k \in Z \mid k > 0\}$. A set $E \subset G$ is independent if the equation $k_1 e_1 + \cdots + k_n e_n = 0$, $k_i \in Z$, $e_i \in E$, yields $k_i e_i = 0$ for all $i = 1, \cdots, n$ or the e_i's are not distinct. Note that an independent set may contain 0. We set $k \times E = \{ke \mid e \in E\}$, and $nE = \{e_1 + \cdots + e_n \mid e_i \in E\}$ for $k \in Z$, $n \in Z^+$; $E_1 + E_2 = \{e_1 + e_2 \mid e_1 \in E_1, e_2 \in E_2\}$. $M(G)$ is the set of bounded regular Borel measures on G.

Here is our main result.

THEOREM. *If G is a locally compact abelian group, $E \subset G$ a compact independent subset, $\mu_1, \cdots, \mu_{n+1} \in M(G)$ positive continuous measures, $k_i \in Z$, $i = 1, \cdots, n$, and $x_0 \in G$, then*

$$\mu_1 * \cdots * \mu_{n+1}(k_1 \times E + \cdots + k_n \times E - x_0) = 0.$$

At the end of the proof of the theorem, we will derive the following corollary:

COROLLARY. *If G is an l.c.a. group, $H \subset G$ is a closed nondiscrete subgroup with Haar measure h, $x_0 \in G$, and $E \subset G$ is compact independent, then $h[(GpE - x_0) \cap H] = 0$, where $GpE = \bigcup_{n=1}^{\infty} n(E \cup -E)$.*

Received by the editors June 12, 1973.

AMS (MOS) subject classifications (1970). Primary 43A05.

Key words and phrases. Compact independent set, Haar measure.

[1] This work formed a part of the author's Ph.D. dissertation, which was written at Northwestern University under the supervision of Colin C. Graham.

2. **Proof of Theorem.** The proof is by induction. If $n=1$, we follow the proof of [2, Theorem 2]. Suppose $k_1=1$. By definition,

$$\mu_1 * \mu_2(E) = \int_G \mu_1(E-x)\, d\mu_2(x). \tag{1}$$

For $x_1 \neq x_2$, suppose there is a fixed element $y_0 \in E-x_1 \cap E-x_2$; then $y_0=e_0-x_1=f_0-x_2$, where $e_0, f_0 \in E$ are fixed. Hence,

$$x_1 + y_0 = e_0; \tag{2.1}$$

$$x_2 + y_0 = f_0. \tag{2.2}$$

If $y \in E-x_1 \cap E-x_2$ is arbitrary, then $y=e-x_1=f-x_2$ for certain $e, f \in E$, so

$$x_1 + y = e; \tag{2.3}$$

$$x_2 + y = f. \tag{2.4}$$

The equation (symbolically written) (2.1)−(2.2)=(2.3)−(2.4) yields $e_0-f_0=e-f$. Since either e_0 or f_0 is nonzero (otherwise $x_1=x_2$), the independence of E requires one of the following four possibilities: $e_0=e$, $e_0=f$, $f_0=e$, or $f_0=f$. By (2.3) and (2.4), these imply

$$y = e_0 - x_1,\ y = f_0 - x_1,\ y = e_0 - x_2, \text{ or } y = f_0 - x_2$$

respectively. Since e_0, f_0, x_1, and x_2 are fixed, this shows

$$\operatorname{card}[E-x_1 \cap E-x_2] \leqq 4$$

and $\mu_1(E-x_1 \cap E-x_2)=0$, by continuity of μ_1. (Since $e_0-x_1=f_0-x_2=y_0$, we actually have card$\leqq 3$.) The set of x with $\mu_1(E-x)>0$ is then countable, since it is easy to see that otherwise $\|\mu_1\|=\infty$. Hence by (1) and the continuity of μ_2, we have $\mu_1 * \mu_2(E)=0$. If $x_0 \in G$, $\mu_1 * \mu_2(E-x_0)=[(\delta_{x_0} * \mu_1) * \mu_2]=0$, where δ_{x_0} is the point mass at x_0. Finally, in the case $k_1 \neq 1$, $k_1 \times E$ is also compact independent, so the Theorem holds for $n=1$.

We now suppose the Theorem holds for $n-1$ and all $k_1, \cdots, k_{n-1} \in Z$; we show it holds for n. Let $\mu_1, \cdots, \mu_{n+1}$, $k_1, \cdots, k_n$ and x_0 be given. Suppose

$$\mu_1 * \cdots * \mu_{n+1}(k_1 \times E + \cdots + k_n \times E) = \delta > 0. \tag{3}$$

LEMMA. *E may be written as the finite disjoint union of Borel sets E_i, $i=1, \cdots, p$, such that for some choice of integers $i_1, \cdots, i_n$, $0 \leqq i_1 < \cdots < i_n \leqq p$, we have*

$$\mu_1 * \cdots * \mu_{n+1}(k_1 \times E_{i_1} + \cdots + k_n \times E_{i_n}) > 0. \tag{4}$$

PROOF OF LEMMA. We first set $k=\sum_1^n |k_i|$. By the inductive hypothesis of the Theorem, for each choice of integers $l_1, \cdots, l_{n-1}$, we have

$$\mu_1 * \cdots * (\mu_n * \mu_{n+1})\left(\sum_1^{n-1} l_i \times E\right) = 0.$$

Containing each set $\sum_1^{n-1} l_i \times E$ there is, therefore, by the regularity of $\mu_1 * \cdots * \mu_{n+1}$, an open set $U_{l_1,\ldots,l_{n-1}}$ with $\sum l_i \times E \subset U_{l_1,\ldots,l_{n-1}}$, and

$$\mu_1 * \cdots * \mu_{n+1}(U_{l_1,\ldots,l_{n-1}}) < \delta/A, \tag{5}$$

where A is the number of possible choices of $l_1, \cdots, l_{n-1}$ with $\sum_1^{n-1} |l_i| \leqq k$. (For example, $A \leqq (2k+1)^{n-1}$.)

For each $U_{l_1,\cdots,l_{n-1}}$, there is a relatively compact neighborhood V of 0 such that

$$\sum_1^{n-1} l_i \times E + k\bar{V} \subset U_{l_1,\cdots,l_{n-1}}, \tag{6}$$

by compactness. Taking (finite) intersections when necessary, we may assume (6) holds with one V for all choices of $(l_1, \cdots, l_{n-1})$ with $\sum_1^{n-1} |l_i| \leqq k$.

Since E is compact, it is covered by finitely many (say p) neighborhoods $(e_i+V)\cap E$, with $e_i \in E$, $i=1, \cdots, p$ (p depending on V). Define $E_1=(e_1+\bar{V})\cap E$,

$$E_i = (e_i + \bar{V}) \cap E \backslash (E_1 \cup \cdots \cup E_{i-1}), \qquad i = 2, \cdots, p,$$

so the E_i's are disjoint Borel sets with union E. Each set of the form

$$F = k_1 \times E_{i_1} + \cdots + k_n \times E_{i_n}, \qquad 1 \leqq i_1, \cdots, i_n \leqq p, \tag{7}$$

is Borel. We claim that at least one of the sets of form (7), with $i_1 < \cdots < i_n$ all distinct, has strictly positive $\mu_1 * \cdots * \mu_{n+1}$-measure. This will prove the lemma.

Indeed, consider any set of form (7) with at least two subscripts equal, say $i_1 = i_2$. It may then be written

$$\begin{aligned} F &= k_1 \times E_{i_1} + k_2 \times E_{i_1} + k_3 \times E_{i_3} + \cdots + k_n \times E_{i_n} \\ &\subset k_1 \times (e_{i_1} + \bar{V}) + k_2 \times (e_{i_1} + \bar{V}) + k_3 \times (e_{i_3} + \bar{V}) \\ &\quad + \cdots + k_n \times (e_{i_n} + \bar{V}) \\ &\subset (k_1 + k_2) \times E + \cdots + k_n \times E + k\bar{V} \\ &\subset U_{(k_1+k_2,\, k_3,\cdots,k_n)} \end{aligned}$$

by (6). Hence the union of such sets (with two subscripts equal) is contained in the union of the (A) sets $U_{l_1,\cdots,l_{n-1}}$, and thus has $\mu_1 * \cdots * \mu_{n+1}$-measure less than δ, by (5). This completes the proof of the Lemma.

For convenience we will assume, relabelling if necessary, that $i_1 = 1, \cdots, i_n = n$. We write

$$\mu_1 * \cdots * \mu_{n+1}\left(\sum_1^n k_i \times E_i\right) = \int \mu_1 * \cdots * \mu_n\left(\sum k_i \times E_i - x\right) d\mu_{n+1}(x). \tag{8}$$

We claim that

$$x_1 \neq x_2 \Rightarrow \mu_1 * \cdots * \mu_n\left(\sum_1^n k_i \times E_i - x_1 \cap \sum_1^n k_i \times E_i - x_2\right) = 0.$$

Then as in the proof of the case $n=1$, we have $\mu_1 * \cdots * \mu_n(\sum k_i \times E_i - x) = 0$ except for perhaps countably many x, and we obtain, by (8),

$$\mu_1 * \cdots * \mu_{n+1}\left(\sum_1^n k_i \times E_i\right) = 0, \tag{9}$$

contradicting (4) and hence (3). We will thus obtain

$$\mu_1 * \cdots * \mu_{n+1}\left(\sum_1^n k_i \times E\right) = 0. \tag{10}$$

To prove the claim, suppose there is a fixed $y_0 \in (\sum k_i \times E_i - x_1) \cap (\sum k_i \times E_i - x_2)$. Then

$$x_1 + y_0 = k_1 e_1^0 + \cdots + k_n e_n^0, \tag{11.1}$$

$$x_2 + y_0 = k_1 f_1^0 + \cdots + k_n f_n^0, \tag{11.2}$$

$e_i^0, f_i^0 \in E_i$ fixed. Given an arbitrary $y \in (\sum k_i \times E_i - x_1) \cap (\sum k_i \times E_i - x_2)$, we have

$$x_1 + y = k_1 e_1 + \cdots + k_n e_n, \tag{11.3}$$

$$x_2 + y = k_1 f_1 + \cdots + k_n f_n, \tag{11.4}$$

$e_i, f_i \in E_i$. For at least one value of i, say $i=j$, we must have $k_j e_j \neq k_j f_j$, since otherwise $x_1 = x_2$. Hence the equation denoted by $(11.1)-(11.2) = (11.3)-(11.4)$ and the disjointness of the E_i's with the independence of E yields

$$k_j e_j^0 - k_j f_j^0 - k_j e_j + k_j f_j = 0. \tag{12}$$

Since one of the first two terms of (12) must be nonzero, we must have (again by independence) that one of e_j or f_j must equal e_j^0 or f_j^0, the latter two being fixed. Then by (11.3) and (11.4), either $y = k_j e_j^0 + \sum_{i \neq j} k_i e_i - x_1$, $y = k_j f_j^0 + \sum_{i \neq j} k_i e_i - x_1$, $y = k_j e_j^0 + \sum_{i \neq j} k_i f_i - x_2$, or $y = k_j f_j^0 + \sum_{i \neq j} k_i f_i - x_2$;

hence we have

$$y \in \bigcup_{s=1,2}\left(k_j e_j^0 + \sum_{i\neq j} k_i \times E_i - x_s\right) \cup \bigcup_{s=1,2}\left(k_j f_j^0 + \sum_{i\neq j} k_i \times E_i - x_s\right).$$

Since y is arbitrary, $(\sum k_i \times E_i - x_1) \cap (\sum k_i \times E_i - x_2)$ is contained in a finite union (over $j=1, \cdots, n$) of sets of the above form; however, by the inductive hypothesis,

$$\mu_1 * \cdots * \mu_{n+1}\left(k_j e_j^0 + \sum_{i\neq j} k_i \times E_i - x_s\right)$$
$$= (\mu_1 * \mu_2) * \cdots * \mu_{n+1}\left(\sum_{i\neq j} k_i \times E_i - (x_s - k_j e_j^0)\right) = 0,$$

(there are n-convolutions and $n-1$-summands); $s=1, 2$, and similarly for f_j^0. Hence the claim, and (10), are proved. (We note it is not generally possible to have $\operatorname{card}[\sum k_i \times E_i - x_1 \cap \sum k_i \times E_i - x_2] < \infty$; for example, choose E infinite and fix $x_1, x_2 \in E$; then $E \subset E+E-x_1 \cap E+E-x_2$.)

Finally,

$$\mu_1 * \cdots * \mu_{n+1}\left(\sum_{i=1}^{n} k_i \times E - x_0\right)$$
$$= (\delta_{x_0} * \mu_1) * \cdots * \mu_{n+1}\left(\sum_{i=1}^{n} k_i \times E - x_0\right) = 0,$$

and the Theorem is proved. Q.E.D.

Note that the requirement that μ_i is positive may be lifted by passing to the total variation measure.

If G is a compact nondiscrete group with Haar measure h, $E \subset G$ compact independent, and $Gp(E) = \bigcup_{n=1}^{\infty} n(E \cup -E)$, then our Theorem implies at once that $h(GpE)=0$. Indeed, h is a bounded, continuous, idempotent measure, and thus annihilates $\sum_{i=1}^{n} k_i \times E$ for all n and k_i. Our Corollary is a generalization of this, due to Graham [**1**] (who strengthened Rudin [**3**, 5.3.6]), giving a proof considerably simpler than Graham's.

3. **Proof of Corollary.** Given a closed set $S \subset G$, define $h|_S(A) = h(A \cap S \cap H)$ for all $A \subset G$, A Borel. In particular, we may write $h|_H = h$, and thus consider h as a (possibly unbounded) measure on G. We claim that if $S \subset H$ and $A \subset G$ are compact sets, then setting $B = A - nS$ for any $n \in Z^+$, we obtain

$$h|_B * h|_S * \cdots * h|_S(A) = h(S)^n h(A), \tag{13}$$

where there are n-convolutions of $h|_S$. Indeed,

$$h|_B * h|_S * \cdots * h|_S(A)$$
$$= \int h|_B(A - x_1 - \cdots - x_n)\, dh|_S(x_1) \cdots dh|_S(x_n).$$

Since $A - x_1 - \cdots - x_n \subset A - nS = B$ for $x_i \in S$, $i = 1, \cdots, n$,

$$h|_B(A - x_1 - \cdots - x_n) = h(A - x_1 - \cdots - x_n) = h(A)$$

because $x_i \in S \subset H$ implies $(A - x_1 - \cdots - x_n) \cap H = A \cap H - x_1 - \cdots - x_n$. Thus,

$$\int h|_B(A - x_1 - \cdots - x_n)\, dh|_S(x_1) \cdots dh|_S(x_n)$$
$$= \int h(A)\, dh|_S(x_1) \cdots dh|_S(x_n)$$
$$= h(S)^n h(A),$$

and the claim is proved.

Suppose now that E and $k_1, \cdots, k_n$ are given; we fix $W \subset H$ to be any relatively compact neighborhood in H, and set $A = k_1 \times E + \cdots + k_n \times E - x_0$, $S = \overline{W}$, $B = A - nS$ as above. Since $h|_B$ and $h|_S$ are then bounded continuous measures on G, our Theorem implies

$$h|_B * h|_S * \cdots * h|_S(A) = 0,$$

where there are n-convolutions of $h|_S$. However, (13) implies that $h(S)^n h(A) = 0$. Since $\overline{W} = S$, then $h(S) \neq 0$ and it follows that $h(A) = 0$. Thus $h[(GpE - x_0) \cap H] = 0$. Q.E.D.

I would like to express my gratitude to the referee for his helpful remarks and criticisms.

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