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A COINCIDENCE THEOREM RELATED TO THE BORSUK-ULAM THEOREM

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ABSTRACT. A coincidence theorem generalizing the classical result of Borsuk on maps of S^n into R^n is proved, in which the antipodal map is replaced by a Z_p-action on a space which is $(n-1)(p-1)$-connected.

The main result is:

THEOREM 1. *Let X be a Hausdorff space which supports a free Z_p-action, and $f:X\to R^n$ a continuous map, $n\geqq 2$. If X is $(n-1)(p-1)$-connected, then there exists $x\in X$ and $g\in Z_p$, $g\neq$ identity, such that $f(x)=f(gx)$.*

We observe that if $p=2$, then Theorem 1 is a restatement of the classical Borsuk-Ulam theorem.

The case $n=2$ has been studied by the second author [**3**], using the fact that Artin's braid groups have no elements of finite order. For this case it suffices to assume only that $\pi_1(X)$ is a torsion group.

The cases $n>2$ require a bit more geometry. We recall the definition of the configuration space $F(M,j)$, of j distinct points in a space M: $F(M,j)$ is the subspace of M^j given by $\{\langle x_1,\cdots,x_j\rangle \mid x_i\in M,\ x_i\neq x_j$ if $i\neq j\}$. The spaces $F(M,j)$ have been studied by Fadell and Neuwirth [**4**]. Evidently Σ_j, the symmetric group on j letters, acts freely on $F(M,j)$ by permutation of coordinates.

We define $F(R^\infty,j)$ to be inj $\lim_n F(R^n,j)$, where $F(R^n,j)\subset F(R^{n+1},j)$ is given by the standard inclusion of R^n in R^{n+1}. By [**2**], $F(R^\infty,j)$ is contractible. Since Z_p, the cyclic group of order p, acts on $F(R^n,p)$ and $F(R^\infty,p)$ via the action given by a homomorphism $Z_p\to\Sigma_p$ which sends $1\in Z_p$ to the cycle $(1,2,\cdots,p)$, it follows that $F(R^\infty,p)/Z_p$ is a $K(Z_p,1)$-space. We shall assume without loss of generality that p in the hypothesis of Theorem 1 is prime.

With these preliminaries, we state the main lemma; the lemma's proof is deferred till after the proof of Theorem 1.

LEMMA 2. *$H^i(F(R^n,p)/Z_p;Z_p)=0$ if $i>(n-1)(p-1)$.*

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PROOF OF THEOREM 1. Let σ denote the generator of the cyclic group Z_p. We suppose that Theorem 1 is false, i.e., $f(x)\neq f(\sigma^i x)$ for all $x\in X$ and all i such that $1\leqq i\leqq p-1$. Define $\psi: X\to F(R^n, p)$ by the formula $\psi(x)=\langle f(x), f(\sigma x), \cdots, f(\sigma^{p-1}x)\rangle$. Clearly ψ is a continuous, Z_p-equivariant map. Since X is Hausdorff, ψ induces a map of covering spaces:

$$\begin{tikzcd}
X \arrow[r, "\psi"] \arrow[d] & F(R^n, p) \arrow[r, "\lambda"] \arrow[d] & F(R^\infty, p) \arrow[d] \\
X/Z_p \arrow[r, "\hat{\psi}"] & F(R^n, p)/Z_p \arrow[r, "\hat{\lambda}"] & F(R^\infty, p)/Z_p
\end{tikzcd}$$

(vertical arrows represent quotient maps; λ and $\hat{\lambda}$ are the obvious inclusions). By naturality of the spectral sequence for a covering [**1**, pp. 355–358] and the fact that X is $(n-1)(p-1)$-connected, it follows immediately that $\hat{\psi}^*\circ\hat{\lambda}^*$: $H^*(K(Z_p, 1); Z_p)\to H^*(X/Z_p; Z_p)$ is an isomorphism in degrees $\leqq(n-1)(p-1)$ and a monomorphism in degree $(n-1)(p-1)+1$.

It is well known that

$$\begin{aligned} H^*(K(Z_p, 1); Z_p) &= P[u] \quad \text{if } p=2, \\ &= E[u]\otimes P[\beta u] \quad \text{if } p>2, \end{aligned}$$

as an algebra, where $P[u]$ denotes the polynomial algebra on a one-dimensional class u, $E[u]$ denotes the exterior algebra on a one-dimensional class u, and $P[\beta u]$ denotes the polynomial algebra on the Bockstein of u [**1**, p. 252]. (We will consider the cases $p>2$ since the case $p=2$ is analogous and easier.) Hence $\hat{\psi}^*\circ\hat{\lambda}^*(u^\varepsilon(\beta u)^k)\in H^*(X/Z_p; Z_p)$ is nonzero provided $\varepsilon=0$, 1 and $\varepsilon+2k\leqq(n-1)(p-1)+1$. But by Lemma 2, $\hat{\lambda}^*(u^\varepsilon(\beta u)^k)=0$ if $\varepsilon+2k=(n-1)(p-1)+1$, which is a contradiction to our hypothesis that $f(x)\neq f(\sigma^i x)$ for $x\in X$, $1\leqq i\leqq p-1$. This proves Theorem 1.

We remark that this proof of Theorem 1 is actually a generalization of the proof of the Borsuk-Ulam theorem which relies on the truncated polynomial algebra $H^*(P^n; Z_2)$.

PROOF OF LEMMA 2. Let $\{E_r\}$ denote the spectral sequence for the covering whose E_2^{**} term is $H^*(Z_p; H^*(F(R^n, p); Z_p))$ and which converges to $H^*(F(R^n, p)/Z_p; Z_p)$. By Theorem IV of [**2**] (the 'vanishing theorem'), $E_2^{s,t}=0$ if $s>0$ and $t\neq 0$, $(n-1)(p-1)$ or $t>(n-1)(p-1)$. By the periodicity argument in [**2**], it is easy to see that no classes of total degree greater than $(n-1)(p-1)$ can survive to E_∞^{**}. This proves Lemma 2.

REMARKS. Another generalization of the Borsuk-Ulam Theorem has been proved by Munkholm [**5**], whose result implies that if $f: S^k\to R^n$ is continuous and $\sigma: S^k\to S^k$ generates a Z_p-action on S^k and $k\geqq n(p-1)$, then there exists $x \in S^k$ such that $f(x)=f(\sigma^i x)$ for *all* i, $1 \leqq i \leqq p-1$. Thus Munkholm's result requires a stronger hypothesis than our theorem, but also yields a stronger conclusion.

For $p \geqq 3$ and $n \geqq 1$, one easily finds continuous maps $f: S^{2n-1} \to R^{n+1}$ for which $f(x) \neq f(\sigma^i x)$ for any i, $1 \leqq i \leqq p-1$, and any $x \in S^{2n-1}$. This shows that Theorem 1 is best possible for $p=3$, in the sense that the hypothesis that X is $(n-1)(p-1)$-connected cannot be weakened.

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