PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 1, May 1974

THE RING OF HOLOMORPHIC FUNCTIONS ON A STEIN COMPACT SET AS A UNIQUE FACTORIZATION DOMAIN

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ABSTRACT. Let Γ be the ring of germs of analytic functions on a Stein compact subset K of a complex-analytic space. Necessary and sufficient conditions on K for Γ to be a unique factorization domain are given.

Let X be a complex-analytic space with structure sheaf $\mathcal{O}$, and let K be a compact Stein subset of X, so that K has a neighbourhood base in X consisting of Stein open subsets of X. Let $\Gamma=\Gamma(K, \mathcal{O})$, the ring of germs of analytic functions on K. In this note, we determine necessary and sufficient conditions for Γ to be a unique factorization domain.

Conditions for Γ to be a Noetherian ring have been given by Siu [**5**], following an earlier result of Frisch [**2**]:

THEOREM (SIU). *Let K be a compact Stein subset of an analytic space $(X, \mathcal{O})$. Then $\Gamma(K, \mathcal{O})$ is Noetherian if and only if $V\cap K$ has only finitely many topological components for each complex-analytic subvariety V defined in an open neighbourhood of K.*

If z is a point of an arbitrary complex-analytic space, then $\mathcal{O}_z$, the ring of analytic function germs at z, is not necessarily a unique factorization domain, for, if it is, then z is a normal point of X. On the other hand, if z is a simple point of X, which means that $\mathcal{O}_z$ is a regular local ring, or, equivalently, that a neighbourhood of z in X can be mapped bianalytically onto an open set in C^n, then $\mathcal{O}_z$ is a unique factorization domain by [**4**, §4.2]. We shall suppose explicitly that $\mathcal{O}_z$ is a unique factorization domain for each z in K. Thus, the hypotheses on K are satisfied if K is a compact subset of a complex manifold X and K is the intersection of a sequence of open Stein manifolds (i.e., K is a holomorphic set).

We shall require Cartan's theorems A and B for coherent analytic sheaves over the compact set K. These theorems are stated for the context in which we require them in [**1**, Théorème 1].

Received by the editors April 10, 1973 and, in revised form, July 27, 1973.

AMS (MOS) subject classifications (1970). Primary 32E25; Secondary 13E05, 13F15.

Key words and phrases. Analytic space, germs of analytic functions, Stein compact set, Noetherian ring, unique factorization domain.

Let $\mathscr{M}$ denote the sheaf of meromorphic functions on K, and let $\mathcal{O}^*$ denote the sheaf of invertible elements in $\mathcal{O}$ (with multiplication as group operation). We say that K is a *Cousin* II *set* if the second Cousin problem can be solved on K: if $\{U_\alpha\}$ is a cover of K consisting of (relatively) open subsets, and if $f_\alpha \in \Gamma(U_\alpha, \mathcal{O})$ such that $f_\alpha f_\beta^{-1} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}^*)$ for each α, β, then there exists $f \in \Gamma(K, \mathcal{O})$ such that $f f_\alpha^{-1} \in \Gamma(U_\alpha, \mathcal{O}^*)$ for each α. The set K is a *strong Poincaré set* if, given $m \in \Gamma(K, \mathscr{M})$, there exist $f, g \in \Gamma(K, \mathcal{O})$ such that $m = fg^{-1}$ and $(f_z, g_z) = 1_z$ in $\mathcal{O}_z$ $(z \in K)$ (i.e., the germs f_z and g_z are coprime for each z). Let I be an ideal in $\Gamma(K, \mathcal{O})$, and let $\mathscr{I}$ be the sheaf of ideals of $\mathcal{O}|K$ generated over K by I. Then we say that I is a *locally principal ideal* in Γ if $\mathscr{I}_z$ is a principal ideal in $\mathcal{O}_z$ for each $z \in K$.

THEOREM 1. *Let K be a compact Stein subset of a complex-analytic space X with structure sheaf $\mathcal{O}$, and suppose $\mathcal{O}_z$ is a unique factorization domain for each $z \in K$. Suppose that $\Gamma = \Gamma(K, \mathcal{O})$ is a Noetherian domain. Then the following conditions are equivalent*:

(1) Γ *is a unique factorization domain*;

(2) $H^2(K; \mathbf{Z}) = \{0\}$;

(3) K *is a Cousin* II *set*;

(4) K *is a strong Poincaré set*;

(5) *every locally principal ideal in* Γ *is a principal ideal.*

PROOF. We first prove the implications which do not require the supposition that Γ be a Noetherian ring.

(2)$\Leftrightarrow$(3)$\Rightarrow$(4). Using Cartan's Theorems A and B, these are standard results (cf. [**3**, VIII, B]).

(5)$\Rightarrow$(3). Let $\{f_\alpha; U_\alpha\}$ be data for the second Cousin problem on K. Define the sheaf $\mathscr{F}$ over K by the conditions $\mathscr{F}_z = \langle f_{\alpha,z} \rangle$ $(z \in U_\alpha)$. Then $\mathscr{F}_z$ is well defined, for $\langle f_{\alpha,z} \rangle = \langle f_{\beta,z} \rangle$ whenever $z \in U_\alpha \cap U_\beta$, and $\mathscr{F}$ is a coherent analytic sheaf over K. Let $I = \Gamma(K, \mathscr{F})$ be the algebra of sections of $\mathscr{F}$ over K, and let I generate $\mathscr{I}$ in $\mathcal{O}|K$. By Theorem A, $\mathscr{I}_z = \mathscr{F}_z$ for $z \in K$, so that I is a locally principal ideal in Γ. By hypothesis, I is a principal ideal in Γ, say $I = \langle f \rangle$. Clearly, f is a solution of the problem with data $\{f_\alpha; U_\alpha\}$.

(1)$\Rightarrow$(4). Let $m \in \Gamma(K, \mathscr{M})$. Since K is a Stein set, we can write $m = f_1 g_1^{-1}$ with $f_1 g_1 \in \Gamma$ (cf. [**3**, VIII, B, 10]), and, by Theorem A again, we can suppose that $(f_{1,z}, g_{1,z}) = 1_z$ for some fixed $z \in K$. Clearly, using the hypothesis, we can also suppose that $(f_1, g_1) = 1$ in Γ. Let w be any point of K, and write $m = f_2 g_2^{-1}$ with $f_2, g_2 \in \Gamma$, $(f_2, g_2) = 1$ in Γ, and $(f_{2,w}, g_{2,w}) = 1_w$. Then $f_1 g_2 = f_2 g_1$ in Γ, and, since Γ is a unique factorization domain, $f_1 \sim f_2$ and $g_1 \sim g_2$ in Γ. Thus, $(f_{1,w}, g_{1,w}) = (f_{2,w}, g_{2,w}) = 1_w$ in $\mathcal{O}_w$ for each

$w \in K$. It follows that f_1 and g_1 are coprime at each point of K, and we have written $m=f_1g_1^{-1}$ in the required form.

In the proof of the remaining implications, we use the fact that Γ is a Noetherian domain.

(4)$\Rightarrow$(1). Since Γ is Noetherian, it is sufficient to prove that every irreducible element in Γ is prime. Let f be an irreducible in Γ, and suppose that $gh \in \langle f\rangle$, say $gh=ff_1$ in Γ. Since K is a strong Poincaré set and Γ is an integral domain, we may suppose that $(g_z, f_{1,z})=1_z$ in $\mathcal{O}_z$ $(z \in K)$. Since $g_zh_z=f_zf_{1,z}$, $h_z \in \langle f_{1,z}\rangle$ $(z \in K)$, and so, from Theorem B, $h \in \langle f_1\rangle$ in Γ, say $h=f_1h_1$. Thus, $gh_1=f$, and since f is irreducible, either g or h_1 is a unit in Γ. In the former case, $h \in \langle f\rangle$, and in the latter, $g \in \langle f\rangle$, so that f is a prime, as required.

(1)$\Rightarrow$(5). Let I be a locally principal ideal in Γ, and let $\mathscr{I}$ be as above. Since I is finitely generated, $\mathscr{I}$ is a coherent analytic sheaf over K.

Let $z \in K$. By hypothesis, $\mathscr{I}_z$ is a principal ideal in $\mathcal{O}_z$, and so, by Theorem A, there exists $f \in I$ such that $\mathscr{I}_z=\langle f_z\rangle$ in $\mathcal{O}_z$. Since K is compact, there exist $f_1, \cdots, f_k \in I$ such that, for each $z \in K$, $\mathscr{I}_z=\langle f_{i,z}\rangle$ for some $i=i(z) \in \{1, \cdots, k\}$. By Theorem B, $I=\langle f_1, \cdots, f_k\rangle$. Since Γ is a unique factorization domain, $f_1, \cdots, f_k$ have a highest common factor in Γ, say $g=(f_1, \cdots, f_k)$. Let $h_i=f_ig^{-1}$ $(i=1, \cdots, k)$, so that $(h_1, \cdots, h_k)=1$.

We now use hypothesis (1) to prove that, if $(p_1, \cdots, p_n)=1$ in Γ, then $(p_{1,z}, \cdots, p_{n,z})=1_z$ in $\mathcal{O}_z$ $(z \in K)$. The result holds for the case $n=2$ by the result '(1)$\Rightarrow$(4)', above, and the general result follows by an immediate inductive argument. Thus, we see that we have $h_1, \cdots, h_k \in \Gamma$ with

$$(*) \qquad (h_{1,z}, \cdots, h_{k,z}) = 1_z \quad \text{in } \mathcal{O}_z \ (z \in K).$$

Take $z \in K$, and suppose that $\mathscr{I}_z=\langle f_{j,z}\rangle$. If $i \in \{1, \cdots, k\}$, there exists $p_{i,z} \in \mathcal{O}_z$ such that $f_{i,z}=p_{i,z}f_{j,z}$. Thus, $h_{i,z}=p_{i,z}h_{j,z}$ and so $h_{j,z}|h_{i,z}$ in $\mathcal{O}_z$ $(i \in \{1, \cdots, k\})$. This shows that $h_{j,z}|(h_{1,z}, \cdots, h_{k,z})$ in $\mathcal{O}_z$. From $(*)$, $h_{j,z}$ is a unit in $\mathcal{O}_z$, and we have proved that the functions $h_1, \cdots, h_k$ have no common zero on K. It is a consequence of Cartan's Theorem B [**3**, VIII, A, 16] that there exist $p_1, \cdots, p_k \in \Gamma$ such that $\sum p_ih_i=1$. Hence, $g=\sum p_igf_i \in I$, $I=\langle g\rangle$, and I is a principal ideal in Γ, as required.

This concludes the proof of the theorem. □

Suppose that Γ is a regular Noetherian domain in the sense of Kaplansky [**4**]. Then the following proof that '(5)$\Rightarrow$(1)' holds.

Let I be an ideal in Γ generating the coherent sheaf $\mathscr{I}$ in $\mathcal{O}|K$. Suppose that I is invertible [**4**, p. 37] in Γ. We first note that $\mathscr{I}_z$ is invertible in $\mathcal{O}_z$ $(z \in K)$. We must prove that $\mathcal{O}_z \subset \mathscr{I}_z\mathscr{I}_z^{-1}$, so take $f_z \in \mathcal{O}_z$. Then $f_z \in \sum f_{i,z}\mathcal{O}_z$ with $f_i \in \Gamma$. Since I is invertible, there exist $g_{ij} \in I$ and $h_{ij} \in I^{-1}$ such that $f_i=\sum_j g_{ij}h_{ij}$ for each i. Thus, $f_z \in \sum_{i,j} g_{ij,z}h_{ij,z}\mathcal{O}_z$. Clearly, $g_{ij,z} \in \mathscr{I}_z$. Also, $h_{ij,z} \in \mathscr{I}_z^{-1}$, for, if $p_z \in \mathscr{I}_z$, then, by Theorem A,

$p_z \in \sum q_{k,z}\mathcal{O}_z$ with $q_k \in I$, and, since I is invertible, $h_{ij}(\sum q_k\Gamma) \subset \Gamma$, so that $h_{ij,z}p_z \in \mathcal{O}_z$ and $h_{ij,z} \in \mathcal{I}_z^{-1}$, as required. Thus, $\mathcal{O}_z \subset \mathcal{I}_z\mathcal{I}_z^{-1}$.

Now, by [**4**, Theorem 60], $\mathcal{I}_z$ is principal in $\mathcal{O}_z$ $(z \in K)$, so, by hypothesis, I is principal in Γ, and the result follows by [**4**, Theorem 185]. □

If K is a compact, holomorphic set in $\boldsymbol{C}^n$ for which Γ is a Noetherian domain, then Γ is a regular Noetherian domain. To show this, it suffices to show that every maximal ideal of Γ can be generated by an R-sequence [**4**, §3.1]. But if M is a maximal ideal of Γ, then $M=\{f \in \Gamma : f(z^0)=0\}$ for some $z^0 \in K$, and it is clear that the elements $z_1-z_1^0, \cdots, z_n-z_n^0$ form the required R-sequence.

COROLLARY. *Let Δ be a compact polydisc in $\boldsymbol{C}^n$. Then $\Gamma(\Delta, \mathcal{O})$ is a unique factorization domain.*

PROOF. That $\Gamma(\Delta, \mathcal{O})$ is a Noetherian domain is noted in [**2**]. Certainly, $H^2(\Delta; \boldsymbol{Z})=\{0\}$.

When $\Gamma=\Gamma(K, \mathcal{O})$ is Noetherian, we have the following Nullstellensatz for Γ. We use the notation of [**3**, II, E].

PROPOSITION 2. *Let K be a compact Stein subset of a complex-analytic space X, and suppose that Γ is a Noetherian domain. Let I be an ideal in Γ. Then* id loc $I=$ rad I.

PROOF. Certainly, rad $I \subset$ id loc I. Take $f \in$ id loc I. If $z \in K$, then there exists $k(z) \in N$ such that $f_z^{k(z)} \in \mathcal{I}_z$. This is the standard Nullstellensatz, proved in the local complex-analytic case in [**3**, III. A. 7]; the general case follows by writing $\mathcal{I}$ locally as a quotient of an ideal sheaf in $\boldsymbol{C}^n$ [**6**, Lemma]. For w in a neighbourhood of z, $f_w^{k(z)} \in \mathcal{I}_w$. Take a finite refinement of the neighbourhoods covering K, corresponding to $z_1, \cdots, z_m$, and let $k=\max\{k(z_1), \cdots, k(z_m)\}$. Then $f_z^k \in \mathcal{I}_z$ $(z \in K)$, and, since $\mathcal{I}$ is coherent, $f^k \in I$. Thus, id loc $I \subset$ rad I, as required. □

Let $f \in \Gamma$, and write $V(f)=\{z \in K : f(z)=0\}$. A variety V is irreducible if $V=V_1 \cup V_2$, where V_1 and V_2 are varieties, implies that either $V=V_1$ or $V=V_2$.

THEOREM 3. *Let K and X be as in Theorem 1. Suppose that Γ is a Noetherian domain which is a unique factorization domain, and let $f \in \Gamma$.*

(i) *$V(f)$ is an irreducible variety if and only if $f=g^n$, where g is irreducible in Γ.*

(ii) *If $f=\prod_{i=1}^n f_i^{k_i}$ is the factorization of f into irreducible factors in Γ, then $V(f)=\bigcup_{i=1}^n V(f_i)$ is the decomposition of V into its irreducible branches.*

PROOF. If $f \in \Gamma$ is irreducible, then $\langle f\rangle$ is prime, and so, by Proposition 2, id $V(f)$ is prime. The results now follow by the same arguments as those of [**3**, II, E].

I am grateful to Dr. P. F. Smith and Professor E. L. Stout for valuable conversations on the above results.

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