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ESTIMATES FOR THE IMAGINARY PARTS OF THE ZEROS OF A POLYNOMIAL

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ABSTRACT. Estimates for the imaginary parts of the zeros of a polynomial are obtained as a function of its L^2 norm on an interval with respect to an arbitrary weight. The resulting inequality is sharp. Orthogonal polynomials with respect to the given weight are used as the main tool. Simple consequences are deduced and L^p norms are also considered.

1. Some twenty years ago, P. Turán [9], [10] started the study of the location of the zeros of a polynomial given its development in terms of a set of polynomials orthogonal on an interval of the real axis. This turned out to be specially useful for dealing with the imaginary parts of the zeros and was carried further by other people, notably by W. Specht [5], [6], [7], [8]. It is the purpose of this note to establish some results of this type; they are refinements of inequalities due to Specht.

2. Let $\omega(x)$ be a nonnegative function on an interval (a, b) such that all the integrals

$$\Omega_m = \int_a^b x^m \omega(x)\, dx \qquad (m = 0, 1, 2, \cdots)$$

exist and $\Omega_0 > 0$. For $m = 0, 1, 2, \cdots$, let

$$\Psi_m(z) = q_m z^m + q_m^* z^{m-1} + \cdots, \qquad q_m > 0$$

be polynomials orthonormal on (a, b) with respect to the weight function $\omega(x)$. Any polynomial of degree n,

$$f(z) = b_0 + b_1 z + \cdots + b_n z^n \tag{1}$$

can be expanded in terms of the orthonormal polynomials $\Psi_m(z)$:

$$f(z) = a_0 \Psi_0(z) + a_1 \Psi_1(z) + \cdots + a_n \Psi_n(z). \tag{2}$$

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It was proved by Specht [5, p. 363] that all the zeros of $f(z)$ lie in the strip

$$|\operatorname{Im} z| \leqq \frac{q_{n-1}}{q_n}\left(\sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right|^2\right)^{1/2}$$

We prove the following theorem which sharpens this result.

THEOREM 1. *Let $\zeta_1, \zeta_2, \cdots, \zeta_n$ be the zeros of the polynomial* (2). *Then*

$$\sum_{k=1}^{n} |\operatorname{Im} \zeta_k| \leqq \frac{q_{n-1}}{q_n}\left(\sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right|^2\right)^{1/2} \tag{3}$$

with equality if and only if $a_0=a_1=\cdots=a_{n-2}=0$ and $\operatorname{Re}(a_{n-1}/a_n)=0$.

PROOF. We start with the identity

$$\sum_{k=0}^{n} |a_k|^2 = \int_a^b |f(x)|^2\,\omega(x)\,dx = \|f\|^2.$$

In particular, we have $|a_{n-1}|^2+|a_n|^2 \leqq \|f\|^2$. Now

$$f(z) = a_n q_n \prod_{k=1}^{n} (z-\zeta_k) = a_n q_n\left(z^n - \left(\sum_{k=1}^{n}\zeta_k\right) z^{n-1} + \cdots\right)$$

and

$$f(z) = \sum_{k=0}^{n} a_k \Psi_k(z) = a_n q_n z^n + (a_n q_n^* + a_{n-1} q_{n-1}) z^{n-1} + \cdots,$$

so that

$$a_n q_n^* + a_{n-1} q_{n-1} = -a_n q_n \sum_{k=1}^{n} \zeta_k. \tag{4}$$

It is sufficient to prove the theorem when $a_n=1$. In that case, since q_n^* is real, we have $\operatorname{Im} a_{n-1} = -(q_n/q_{n-1}) \sum_{k=1}^{n} \operatorname{Im} \zeta_k$. Hence

$$\frac{q_n}{q_{n-1}}\left|\sum_{k=1}^{n} \operatorname{Im} \zeta_k\right| = |\operatorname{Im} a_{n-1}| \leqq |a_{n-1}| \leqq (\|f\|^2 - 1)^{1/2},$$

so that

$$1 + \left(\frac{q_n}{q_{n-1}}\right)^2 \left|\sum_{k=1}^{n} \operatorname{Im} \zeta_k\right|^2 \leqq \|f\|^2.$$

Let us apply this result to the polynomial $g(z)=f(z)\prod_\nu (z-\bar{\zeta}_\nu)/(z-\zeta_\nu)$ where the zeros ζ_ν appearing in the product are precisely those for which $\operatorname{Im} \zeta_\nu<0$. Since $\|f\|=\|g\|$, we get

$$1 + \left(\frac{q_n}{q_{n-1}}\right)^2 \left(\sum_{k=1}^{n} |\operatorname{Im} \zeta_k|\right)^2 \leqq \|f\|^2$$

which is the statement of the theorem (when $a_n=1$). The proof shows that equality will hold in (3) if and only if $a_0=a_1=\cdots=a_{n-2}=0$, a_{n-1}/a_n is purely imaginary and the zeros $\zeta_1, \zeta_2, \cdots, \zeta_n$ of $f(z)$ are either all above or all below the real axis. It is a remarkable fact (see [**3**, p. 21]) that the zeros of $f(z)=\Psi_n(z)+ic\Psi_{n-1}(z)$ satisfy this last condition for any real number c.

3. From (3), we get the following result for the zeros of the derivative of $f(z)$:

COROLLARY 1. *Let $\xi_1, \xi_2, \cdots, \xi_{n-1}$ be the zeros of the derivative of the polynomial* (2). *Then*

$$\sum_{k=1}^{n-1} |\operatorname{Im} \xi_k| \leqq \frac{n-1}{n} \frac{q_{n-1}}{q_n} \left(\sum_{k=0}^{n-1} \left| \frac{a_k}{a_n} \right|^2 \right)^{1/2}$$

with equality if and only if $f(z)$ is a multiple of the polynomial $\Psi_n(z)+ ic\Psi_{n-1}(z)$ with c real.

Indeed, it is known (see [**1**, p. 264]) that

$$\frac{1}{n-1} \sum_{k=1}^{n-1} |\operatorname{Im} \xi_k| \leqq \frac{1}{n} \sum_{k=1}^{n} |\operatorname{Im} \zeta_k|,$$

equality taking place precisely when all the zeros of $f(z)$ are on one side of the real axis.

Another consequence of Theorem 1 is the following

COROLLARY 2. *There is at least one zero of the polynomial* (2) *in the strip*

$$|\operatorname{Im} z| \leqq \frac{1}{n} \frac{q_{n-1}}{q_n} \left(\sum_{k=0}^{n-1} \left| \frac{a_k}{a_n} \right|^2 \right)^{1/2} \tag{5}$$

It is easy to see that unless $f(z)=a_n\Psi_n(z)$, there is strict inequality in (5) for $n\geqq 2$. In fact, suppose that for some real number c, we had

$$P(z) = \Psi_n(z) + ic\Psi_{n-1}(z) = q_n \prod_{k=1}^{n} (z - (\alpha_k + i\beta))$$

with $\alpha_1, \alpha_2, \cdots, \alpha_n$ and β real. Then all the zeros of the polynomial

$$Q(z) = P(z + i\beta)$$

would be real. There would therefore exist a real number x_0 such that $Q^{(n-1)}(x_0)=0$. This would imply that $0=\beta q_n+cq_{n-1}$.

On the other hand, since the coefficient of z^{n-1} in $Q(z)$ would be real, we would also get $0=n\beta q_n+cq_{n-1}$. Hence $\beta=c=0$.

Corollary 2 is equivalent to the following fact; consider all polynomials of the form (1) such that $\int_a^b |f(x)|^2\omega(x)\,dx = M^2$; then, if $f(z)$ does not vanish in the strip $|\text{Im } z| \leqq K$, we have

$$|b_n| \leqq q_n M/(1 + (nq_n K/q_{n-1})^2)^{1/2}.$$

4. Let $\eta_1 < \eta_2 < \cdots < \eta_m$ be the zeros of $\Psi_m(z)$, and let $d_m(\zeta)$ denote the distance of the point ζ from the interval $[\eta_1, \eta_m]$ which is the convex hull of the zeros of $\Psi_m(z)$. It has been proved by Specht that if $\zeta_1, \zeta_2, \cdots, \zeta_n$ are the zeros of the polynomial (2), then

$$\sum_{k=1}^{n} \left(\frac{1}{q_{k-1}} d_n(\zeta_n) d_n(\zeta_{n-1}) \cdots d_n(\zeta_k) \right)^2 \leqq \frac{1}{q_n^2} \sum_{k=0}^{n-1} \left| \frac{a_k}{a_n} \right|^2.$$

We can easily deduce a comparable inequality from Theorem 1.

THEOREM 2. *Let $\zeta_1, \zeta_2, \cdots, \zeta_n$ be the zeros of the polynomial* (2). *Then*

$$\sum_{k=1}^{n} d_n(\zeta_k) \leqq \frac{q_{n-1}}{q_n} \left(\sum_{k=0}^{n-1} \left| \frac{a_k}{a_n} \right|^2 \right)^{1/2} \tag{6}$$

with equality if and only if $a_0 = a_1 = \cdots = a_{n-2} = 0$ and $\text{Re}(a_{n-1}/a_n) = 0$.

PROOF. Let

$$f^*(z) = a_n q_n \prod_{k=1}^{n} (z - \zeta_k^*)$$

where

$$\begin{aligned} \zeta_k^* &= \zeta_k && \text{if } \eta_1 \leqq \text{Re } \zeta_k \leqq \eta_n, \\ &= \eta_n + i\,|\zeta_k - \eta_n| && \text{if Re } \zeta_k > \eta_n, \\ &= \eta_1 + i\,|\zeta_k - \eta_1| && \text{if Re } \zeta_k < \eta_1, \end{aligned}$$

for $k = 1, 2, \cdots, n$. Then, by virtue of Theorem 1, we have

$$\sum_{k=1}^{n} d_n(\zeta_k) = \sum_{k=1}^{n} |\text{Im } \zeta_k^*| \leqq (1/|a_n|)(q_{n-1}/q_n)(\|f^*\|^2 - |a_n|^2)^{1/2},$$

and it remains only to show that $\|f^*\| \leqq \|f\|$. This can be done by using the Gauss quadrature formula with nodes at the points η_k (see for example [2, p. 320]):

$$\int_a^b F(x)\omega(x)\,dx = \sum_{k=1}^{n} H_{k,n} F(\eta_k) + (1/q_n^2) F_{(\eta)}^{(2n)}/(2n)!,$$

where η is some point in (a, b), and the $H_{k,n}$ (the Christoffel numbers) are all positive. In our case $|f(x)|^2$ and $|f^*(x)|^2$ are both polynomials of degree $2n$ in x with their $(2n)$th derivative equal to $(2n)!|a_n|^2 q_n^2$. Moreover, we have $|f^*(\eta_k)| \leqq |f(\eta_k)|$ for $k=1, 2, \cdots, n$. Therefore

$$\|f^*\|^2 = \int_a^b |f^*(x)|^2 \,\omega(x)\,dx = \sum_{k=1}^{n} H_{k,n}\,|f^*(\eta_k)|^2 + |a_n|^2$$
$$\leqq \sum_{k=1}^{n} H_{k,n}\,|f(\eta_k)|^2 + |a_n|^2 = \int_a^b |f(x)|^2\,\omega(x)\,dx = \|f\|^2.$$

Equality will hold in (6) if and only if $f^*(z)$ is of the form $a_n\Psi_n(z) + a_{n-1}^*\Psi_{n-1}(z)$ with $\operatorname{Re}(a_{n-1}^*/a_n)=0$ and $\|f^*\|=\|f\|$. Obviously the last requirement implies that $f^*(z)=f(z)$. This completes the proof of Theorem 2.

Incidentally, this proof yields the following corollary.

COROLLARY 3. *Let*

$$f(x) = (x - x_1)(x - x_2)\cdots(x - x_n),$$
$$g(x) = (x - y_1)(x - y_2)\cdots(x - y_{n-1}),$$

with $x_1 < y_1 < x_2 < \cdots < y_{n-1} < x_n$. *Then, for any real number* c, *the zeros of the polynomial* $h(x)=f(x)+icg(x)$ *are all in the half strip* $\operatorname{Im} z \geqq 0$, $x_1 \leqq \operatorname{Re} z \leqq x_n$, *or all in the conjugate half strip.*

Indeed, $f(x)$ and $g(x)$ are orthogonal on an interval (a, b) containing (x_1, x_n) with respect to an appropriate weight function and the above reasoning applies.

5. It should be noted that other inequalities relating the expressions $\sum_{k=1}^n |\operatorname{Im} \zeta_k|$ and $(\int_a^b |f(x)|^p \omega(x)\,dx)^{1/p}$ can be deduced from the theorem of M. Riesz-Thorin (see [**11**, p. 95]). We now assume that the interval (a, b) is bounded. Let $f(z)$ be a polynomial of the form (2). Then using the inequalities

$$(|a_{n-1}|^2 + |a_n|^2)^{1/2} \leqq \left(\int_a^b |f(x)|^2\,\omega(x)\,dx\right)^{1/2}$$

and

$$|a_{n-1}| + |a_n| \leqq (2\Omega_0)^{1/2} \max_{a \leqq x \leqq b} |f(x)|$$

(recall that $\Omega_0 = \int_a^b \omega(x)\,dx$), it follows from the theorem of M. Riesz-Thorin that, for any $t \in (0, 1)$,

$$(7) \qquad (|a_{n-1}|^{2/(1+t)} + |a_n|^{2/(1+t)})^{(1+t)/2} \leqq (2\Omega_0)^{t/2}\left(\int_a^b |f(x)|^{2/(1-t)}\,\omega(x)\,dx\right)^{(1-t)/2}$$

From this, we infer the following result:

THEOREM 3. *Let* $-\infty<a<b<+\infty$, $2<s<\infty$, *and let* $\zeta_1, \zeta_2, \cdots, \zeta_n$ *be the zeros of*

$$f(z) = z^n + b_{n-1}z^{n-1} + \cdots = (1/q_n)\Psi_n(z) + a_{n-1}\Psi_{n-1}(z) + \cdots.$$

Then

$$\left\{\left(\frac{1}{q_n}\right)^{s/(s-1)} + \left(\frac{1}{q_{n-1}}\sum_{k=1}^{n} |\mathrm{Im}\,\zeta_k|\right)^{s/(s-1)}\right\}^{(s-1)/s} \leqq (2\Omega_0)^{(s-2)/2s}\left(\int_a^b |f(x)|^s\,\omega(x)\,dx\right)^{1/s}. \tag{8}$$

PROOF. Without changing the value of the right-hand side of (8), we can assume that $\mathrm{Im}\,\zeta_k \geqq 0$ for $k=1, 2, \cdots, n$. In the present case $a_n=1/q_n$ and we have, from (4), $(q_n^*/q_n)+a_{n-1}q_{n-1}=-\sum_{k=1}^n \zeta_k$, and consequently

$$|a_{n-1}| \geqq |\mathrm{Im}\,a_{n-1}| = \frac{1}{q_{n-1}}\sum_{k=1}^{n} |\mathrm{Im}\,\zeta_k|.$$

Now, using (7) we get

$$\left\{\left(\frac{1}{q_n}\right)^{2/(1+t)} + \left(\frac{1}{q_{n-1}}\sum_{k=1}^{n} |\mathrm{Im}\,\zeta_k|\right)^{2/(1+t)}\right\}^{(1+t)/2} \leqq (2\Omega_0)^{t/2}\left(\int_a^b |f(x)|^{2/(1-t)}\,\omega(x)\,dx\right)^{(1-t)/2}$$

for any $t \in (0, 1)$. This is inequality (8) with $t=(s-2)/2$.

6. In a similar way, the Hausdorff-Young inequality (see [**4**, p. 247]) can be used to prove inequalities like (8) for $1 \leqq s \leqq 2$. Now we require that $|\Psi_n(x)|$ be bounded uniformly with respect to n and x in the orthogonality interval. Consider for example the orthonormalized Tchebycheff polynomials of the first kind, for which $|\Psi_n(x)| \leqq (2/\pi)^{1/2}$ for $n=0, 1, 2, \cdots$ and $-1 \leqq x \leqq 1$. Then the Hausdorff-Young inequality implies that, for the polynomial (2),

$$(|a_{n-1}|^{s/(s-1)} + |a_n|^{s/(s-1)})^{(s-1)/s} \leqq \left(\frac{2}{\pi}\right)^{(2-s)/2s}\left(\int_{-1}^{+1} |f(x)|^s\,(1-x^2)^{-1/2}\,dx\right)^{1/s}$$

for $1 \leqq s \leqq 2$. The following result follows from this in the same manner as (8) follows from (7).

THEOREM 4. *Let* $\zeta_1, \zeta_2, \cdots, \zeta_n$ $(n \geqq 2)$ *be the zeros of* $f(z)=z^n+b_{n-1}z^{n-1}+\cdots+b_0$. *Then, for* $1 \leqq s \leqq 2$,

$$(\pi/2)^{1/s}\left(2^{(1-n)s/(s-1)}+\left(2^{(2-n)}\sum_{k=1}^{n}|\mathrm{Im}\,\zeta_k|\right)^{s/(s-1)}\right)^{(s-1)/s} \leqq \left(\int_{-1}^{+1}|f(x)|^s(1-x^2)^{-1/2}\,dx\right)^{1/s}.$$

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