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RELATIVE SIZE OF THE SHILOV BOUNDARY OF A FUNCTION ALGEBRA

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ABSTRACT. A notion of size for subsets of the spectrum of a function algebra is described, relative to which each open subset of the Shilov boundary has the same size as the spectrum.

Let A be a function algebra; denote its spectrum (space of nonzero, continuous, complex-valued homomorphisms, with the weak*-topology) by Σ_A, and its Shilov boundary by Γ_A. In a topological sense, Γ_A may be much smaller than $\sum_A$; for instance it is easy to construct function algebras for which Γ_A is zero-dimensional, while the dimension of Σ_A is as large as we please (see Stout [**8**, p. 372]). In this note we describe another natural notion of size, an analytic notion derived from the theory of several complex variables, relative to which Γ_A and Σ_A have precisely the same size. This generalizes and simplifies some of the recent results of Huckleberry and Stoll [**3**]. The proof uses techniques similar to the function algebra techniques in [**3**].

We will make use of the A-holomorphic functions introduced by Rickart [**4**], [**5**], [**6**], which may be defined by transfinite induction as follows: Let $A_0=A$, and assume that the class A_ν has been defined for all ordinal numbers $\nu<\mu$. A function f defined on a subset E of Σ_A will belong to the class A_μ if for each $x\in E$ there is a compact neighborhood K of x, such that $f|(K\cap E)$ is the uniform limit of functions which are defined on $K\cap E$ and belong to classes A_ν for $\nu<\mu$. We say that a function h defined on a subset E of Σ_A is A-holomorphic if it is an element of the class A_μ for some ordinal number μ and denote the collection of such functions by $\mathscr{H}_A(E)$; we refer to Rickart [**6**] for general information about A-holomorphic functions (and to Stout [**8**] for general facts concerning function algebras). The fundamental result we use is the following

THEOREM. *Let A be a function algebra with spectrum Σ_A and Shilov boundary Γ_A. Then $\mathscr{H}_A(\Sigma_A)$ is a function algebra with spectrum Σ_A and Shilov boundary Γ_A.*

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If Ω is an open subset of Σ_A, then by an A-holomorphic variety in Ω we mean a closed subset V of Ω such that for each $x \in V$, there are an open set Q containing x and a (not necessarily finite) family $\mathscr{F} \subset \mathscr{H}_A(Q)$ for which $V \cap Q = \{y \in Q : f(y) = 0 \text{ for all } f \in \mathscr{F}\}$. In analogy with the usual notion of thinness in several complex variables (see e.g. Gunning and Rossi [**2**, p. 19]) we say that a subset T of Σ_A is A-thin if there is a sequence $V_1, V_2 \cdots$ of A-holomorphic varieties defined in open sets $\Omega_1, \Omega_2 \cdots$ respectively, such that $T \subset \bigcup_{i=1}^{\infty} V_i$, while $\bigcup_{i=1}^{\infty} V_i$ has no interior in Σ_A. Our result is then as follows.

THEOREM. *If A is a function algebra with spectrum Σ_A and Shilov boundary Γ_A, then no relatively open subset of Γ_A is A-thin in Σ_A.*

In order to establish the theorem, we will use a slight modification of a result of Glicksberg [**1**]; the proof is similar to the one in [**3**].

LEMMA. *Let B be a function algebra. If U is a nonempty relatively open subset of Γ_B, then there is an open subset U' of Σ_B for which*

$$\varnothing \neq U' \cap \Gamma_B \subset U$$

and $f|U' \equiv 0$ for each $f \in B$ having the property that $f|U \equiv 0$.

PROOF. Since U is relatively open, it contains a strong boundary point, x_0 say. We can find a function $g \in B$ such that $g(x_0) = \|g\| = 1$, while $|g(y)| < 1$ for $y \in (\Gamma_B - U)$. Set $\alpha = \max\{|g(y)| : y \in (\Gamma_B - U)\}$, $U' = \{x \in \Sigma_B : |g(x)| > \alpha\}$. Then U' is certainly open, $x_0 \in (U' \cap \Gamma_B)$ and $(U' \cap \Gamma_B) \subset U$.

Now let $f \in B$ such that $f|U \equiv 0$. If $\alpha \neq 0$, then for every positive integer n, set $h_n = \alpha^{-n} g^n f$. It is easy to see that $|h_n(x)| \leqq |f_n(x)|$ for each $x \in \Gamma_B$, so that $\|h_n\| \leqq \|f\|$. If $x \in U'$ then $|g(x)| > \alpha$ and

$$|f(x)| = \alpha^n \, |h_n(x)| \, |g(x)|^{-n} \to 0$$

as $n \to \infty$; i.e. $f(x) = 0$. Finally, if $\alpha = 0$, then $fg(x) = 0$ for each $x \in \Gamma_B$, so that $fg \equiv 0$, and again $f|U' \equiv 0$, which completes the proof.

PROOF OF THE THEOREM. Let $V_1, V_2 \cdots$ be a sequence of A-holomorphic varieties in open sets $\Omega_1, \Omega_2 \cdots$ such that $\bigcup_{i=1}^{\infty} V_i$ contains a relatively open subset of Γ_A; we show that $\bigcup_{i=1}^{\infty} V_i$ has interior in Σ_A. Observe first that the Baire category theorem implies that one of these A-holomorphic varieties, say V_1, contains a nonempty relatively open set $U_1 \subset \Gamma_A$. Let $x_1 \in U_1$; there are an open set Q containing x and a family $\mathscr{F} \subset \mathscr{H}_A(Q)$ such that $Q \cap \Gamma_A \subset U_1$ and $V_1 \cap Q = \{y \in Q : f(y) = 0 \text{ for all } f \in \mathscr{F}\}$. We can choose a compact, A-convex neighborhood K of x_1 with $K \subset Q$. Let A' be the uniform completion of $A|K$; then A' is a function algebra, $\Sigma_{A'} = K$, and Rossi's local maximum modulus principle [7] implies that $\Gamma_{A'} \subset (K \cap \Gamma_A) \cup \partial K$, where ∂K denotes the topological boundary of K in Σ_A.

Set $B=\mathscr{H}_{A'}(K)$; then B is a function algebra, $\Sigma_B=K$, $\Gamma_B=\Gamma_{A'}\subset (K\cap\Gamma_A)\cup\partial K$, and $(h|K)\in B$ for each $h\in\mathscr{H}_A(Q)$. Since K is a neighborhood of x_1, there is a relatively open subset U of Γ_B such that $x_1\in U$, $U\subset Q\cap\Gamma_A\subset U_1$. The Lemma now provides an open subset U' of K which contains x_1 such that $h|U'\equiv 0$ for each $h\in(\mathscr{F}|K)$; thus V_1 contains an open subset of Σ_A, as desired.

To derive the Huckleberry-Stoll result from the above, let X be a compact polynomially convex subset of $\boldsymbol{C}^N$ and let $P(X)$ be the uniform completion on X of the polynomials.

COROLLARY (HUCKLEBERRY-STOLL [3]). *If $V_1, V_2\cdots$ is a sequence of varieties in open subsets of $\boldsymbol{C}^N$, and $\bigcup_{j=1}^\infty V_j$ contains a relatively open subset of $\Gamma_{P(X)}$, then $\bigcup_{j=1}^\infty V_j$ contains a relatively open subset of X.*

PROOF. Observe that if Ω is open in $\boldsymbol{C}^N$ and f is holomorphic on Ω, then $f|(\Omega\cap X)$ is $P(X)$-holomorphic; thus each of $V_1\cap X$, $V_2\cap X,\cdots$ is a $P(X)$-holomorphic variety. The Corollary now follows immediately.

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