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# AN INFINITE COMPLEX AND THE SPECTRAL SEQUENCES FOR COMPLEX COBORDISM AND $K$-THEORY

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ABSTRACT. An infinite complex is given whose complex cobordism spectral sequence has differentials, $d_r$, nonzero for infinitely many positive integers, $r$, but whose complex $K$-theory spectral sequence has $d_r$ nontrivial for only finitely many $r$.

**Introduction.** At the Madison conference in 1970, Landweber posed the following problem [**5**, Problem 3, p. 127]: If $X$ is an infinite complex and if for some $r<\infty$ we have $E_r=E_\infty$ in the spectral sequence $H^*(X;Z)\Rightarrow K^*(X)$, must this also be the case for the spectral sequence $H^*(X;MU^*)\Rightarrow MU^*(X)$? Here $K^*(\ )$ is complex $K$-theory and $MU^*(\ )$ is complex cobordism. If the answer were yes, it was hoped that this would yield information about the complex (co)bordism modules of classifying spaces. The purpose of this note is to construct an infinite complex whose existence settles this question negatively.

**Conventions.** All complexes will be pointed. All cohomology theories are reduced ones. If a cohomology theory is represented by the spectrum $A$, $\{E_r(X;A), d_r(X;A)\}$ will stand for the usual spectral sequence $H^*(X;A^*)\Rightarrow A^*(X)$. Fix the prime $p$. $Z_{(p)}$ is the integers localized at $p$ (the set of rational numbers represented by fractions with donominator relatively prime to $p$).

**Localized cohomology theories.** Quillen has given a multiplicative splitting of $MU^*(\ )\otimes Z_{(p)}$ into a sum of shifted copies of $BP^*(\ )$, Brown-Peterson cohomology [**3**]. $BP^*\simeq Z_{(p)}[x_1,\cdots,x_n,\cdots]$ where the dimension of $x_n$ is $-2(p^n-1)$. Adams [**1**] has shown $K^*(\ )\otimes Z_{(p)}$ is isomorphic to a sum $\sum K^*_\alpha(\ )$ where $\alpha$ ranges over the classes of integers modulo $p-1$. Let $\theta$ be the identity of $Z_{p-1}$. Then $K^*_\theta(\ )$ has products and $K^*_\theta\simeq Z_{(p)}[v,v^{-1}]$ where the dimension of $v$ is $-2(p-1)$. After a dimension shift, $K^*_\alpha(X)$ is additively isomorphic to $K^*_\theta(X)$.

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THE EXAMPLE. We wish to construct a locally finite CW complex $X$ such that $E_r(X; K)=E_\infty(X; K)$ for some integer $r$, but such that there is no integer $s$ such that $E_s(X; MU)=E_\infty(X; MU)$.

LEMMA 1. *To construct such a complex, it suffices to construct a sequence of locally finite pointed* CW *complexes* $\{Y(n): n=1, 2, \cdots\}$ *satisfying the following properties.*

(i) *The* $n-1$ *skeleton of* $Y(n)$ *is the singleton containing the base point.*

(ii) $H^*(Y(n); Z)$ *is completely p-primary torsion for some fixed prime p.*

(iii) $E_{2p}(Y(n); K_\theta)=E_\infty(Y(n); K_\theta)$ *for* $n=1, 2, \cdots$.

(iv) *For all but finitely many natural numbers n, there is an integer* $r=r(n)\geqq n$ *such that* $d_r(Y(n); BP)\not\equiv 0$.

PROOF. Let $X=Y(1)\vee Y(2)\vee\cdots$. Hypothesis (i) ensures that $X$ is locally finite. By (ii), $X$ and the $Y(n)$'s have trivial rational cohomology. When we apply the functions $K^*(\ )$, $K_\theta^*(\ )$, $MU^*(\ )$, and $BP^*(\ )$ to these complexes, the results of **[6]**, **[7]** show that we shall have no nonzero elements of infinite filtration. In particular, wedge axioms will hold with respect to $X=Y(1)\vee Y(2)\vee\cdots$ : spectral sequences for $X$ will decompose into direct sums of the respective spectral sequences for the $Y(n)$'s. So (iii) implies $E_{2p}(X; K_\theta)=E_\infty(X; K_\theta)$. By (ii),

$$K^*(X)\cong K^*(X)\otimes Z_{(p)}\cong \sum K_\alpha^*(X)\cong \sum S^{n_\alpha}K_\theta^*(X);$$

so we conclude that $E_{2p}(X; K)=E_\infty(X; K)$. Similarly, $\{E_r(X; BP); d_r(X; BP)\}$ will be a direct summand of $\{E_r(X; MU); d_r(X; MU)\}$. By (iv), $d_r(X; BP)\not\equiv 0$ for an infinite number of values of $r$. $\square$

We need the following lemma to check the second hypothesis of Lemma 1. $Q_0$ and $Q_1$ are the first two generators of the exterior part of the mod $p$ Steenrod algebra (of dimensions 1 and $2p-1$, respectively).

LEMMA 2. *Let $X$ be a locally finite* CW *complex with*

$$H(H^*(X; Z_p); Q_0)=0.$$

*Then there is an isomorphism*

$$E_{2p}^{*,2q(p-1)}(X; K_\theta)\cong H(Q_0H^*(X; Z_p); Q_1).$$

*In particular, if* $H(H^*(X; Z_p); Q_1)=0$ *also, then* $E_{2p}(X; K_\theta)$ *is trivial and* $K_\theta^*(X)=0$.

PROOF. The hypothesis tells us that very element of $H^*(X; Z_{(p)})$ is of order $p$. Thus the mod $p$ reduction homomorphism $\rho: H^*(X; Z_{(p)})\to H^*(X; Z_p)$ is injective and has image $Q_0H^*(X; Z_p)$. (The hypothesis also implies $K_\theta^*(X)$ has no elements of infinite filtration. Compare **[6]**.)

The lemma follows from the fact that Diagram 3 commutes up to multiplication by a unit of $Z_{(p)}$. The fact is implicit in [2] or may be proved using [4, 2.7 and 4.16]. (In (3), $Q_1$ may be replaced by $Q_0P^1$.) □

$$\begin{array}{ccc}
E_{2p-1}^{*,2q(p-1)}(X;K_\theta) & \xrightarrow{d_{2p-1}(X;K_\theta)} & E_{2p-1}^{*+2p-1,2(q-1)(p-1)}(X;K_\theta) \\
\cong & & \cong \\
H^*(X;Z_{(p)}) & & H^{*+2p-1}(X;Z_{(p)}) \\
\downarrow \rho & & \downarrow \rho \\
H^*(X;Z_p) & \xrightarrow{Q_1} & H^{*+2p-1}(X;Z_p)
\end{array}$$

DIAGRAM 3

COROLLARY 4. *A sufficient (but unnecessary) condition for a locally finite* CW *complex to have acyclic complex K-theory is for*

$$H(H^*(X;Z_p);Q_i) = 0,$$

$i=0$ *and* 1, *for every prime p.* □

There are stable complexes $V(0)=S^0\cup_p e^1$ and $V(1)=S^0\cup_p e^1\cup e^{2p-1}\cup_p e^{2p}$ such that $H^*(V(0);Z_p)\cong E[Q_0]$ and $H^*(V(1);Z_p)\cong E[Q_0,Q_1]$. $V(0)$ is just a formal desuspension of a Moore space, but $V(1)$ exists only for odd primes $p$ [8], [9]. For $p>3$, Smith [8] constructs stable maps $\psi_t:S^{2t(p^2-1)}V(1)\to V(1)$, which realize multiplication by $(x_2)^t$ in

$$BP^*(V(1))\cong S^{2p}BP^*/(p,x_1).$$

Let $V(2,t)$ be the stable cofibre of $\psi_t$ and let $Y(t)=S^{4tp^2}V(2,t)$. (Notice that $Y(t)$ is an "honest" complex.)

PROPOSITION 5. $\{Y(t):t=1,2,\cdots\}$ *satisfies the hypothesis of Lemma* 1 *and so* $X=Y(1)\vee Y(2)\vee\cdots$ *settles Landweber's Problem* 3 *negatively.*

PROOF. The first two hypotheses follow from the construction of $Y(t)$. $H^*(Y(t);Z_p)$ is a free $E[Q_0,Q_1]$ module and so we apply Lemma 2 to see that $0=E_{2p}(Y(t);K_\theta)=E_\infty(Y(t);K_\theta)$.

$H^i(V(2,t);Z)$ is zero except when $i=1, 2p, 2t(p^2-1)+2$, and $2t(p^2-1)+2p+1$; in these dimensions, it is isomorphic to $Z_p$.

$$BP^*(V(2,t))\cong S^qBP^*/(p,x_1,x_2^t)$$

where $q=T+2p+1$ and $T=2t(p^2-1)$.

$$0=BP^{2p+1}(V(2,t))=E_\infty^{q,-T}(V(2,t);BP).$$

Let $k$ be the rank of $BP^{-T+2p-2}$. Then the rank of $BP^{-T}$ is at least $k+1$. Since $H^i(V(2,t);Z)=0$ for $i>q$, all classes in $E_2^{q,-T}(V(2,t);BP)$ are infinite cycles. None of the classes live to infinity, so all $k+1$ summands of this group are hit by differentials. Only two possible nonzero differentials have range $E_r^{q,-T}(V(2,t);BP)$, $r\geqq 2$. They are:

(i) $d_{2p-1}(V(2,t);BP)$ with domain $E_{2p-1}^{T+2,-T+2p-2}(V(2,t);BP)$ of rank $k$ and

(ii) $d_{T+1}(V(2,t);BP)$ with domain $E_{T+1}^{2p,0}(V(2,t);BP)$.

Thus this second differential is nonzero and $\{Y(t)\}$ satisfies the fourth hypothesis of Lemma 1. $\square$

REMARK AND ACKNOWLEDGEMENT. Our first example of a family satisfying the hypotheses of Lemma 1 was the family $\{Y(n)\}$ where $Y(n)=RP(\infty)\wedge\cdots\wedge RP(\infty)$ ($n$ times). The necessary computations, which are too lengthy to warrant reproduction here, grow out of an unfruitful approach to a different problem begun jointly with W. Stephen Wilson.

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