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ON DIRICHLET'S THEOREM AND INFINITE PRIMES

CARTER WAID

ABSTRACT. It is shown that Dirichlet's theorem on primes in an arithmetic progression is equivalent to the statement that every unit of a certain quotient ring $\mathbf{Z}$ of the nonstandard integers is the image of an infinite prime. The ring $\mathbf{Z}$ is the completion of Z relative to the "natural" topology on Z.

1. **Notation.** Throughout this note N shall denote the natural numbers, Z the rational integers, and P the positive primes. We shall follow the approach of Machover and Hirschfeld, [**2**], in our use of nonstandard analysis. Thus U is to be a universal set containing N and $^*U$ will be a comprehensive [**6**, p. 446] enlargement of U. The nonstandard natural numbers $^*N$ can be expressed as $^*N=N\cup N_\infty$ where N_∞ is the set of infinite natural numbers. Similarly, $^*P=P\cup P_\infty$, P_∞ the set of infinite primes.

2. LEMMA. *Let a, b be coprime integers. A necessary and sufficient condition that the sequence $|a+bn|$ $(n\in N)$ contains infinitely many primes is that $|a+bn|$ be an infinite prime for some nonstandard natural number n.*

PROOF. Clear.

3. **Completions of** Z. In a series of papers [**4**], [**5**], [**6**], Robinson derives the results of this section in a more general setting.

Let $\mu=\bigcap n\cdot {}^*Z$ $(n\in N)$. The external ideal μ of $^*Z$ is the monad of 0 for the "natural" topology on Z. It can be characterized both as the set of all nonstandard integers divisible by every nonzero standard integer and as the $^*Z$-ideal generated by numbers of the form $n!$ where n is an infinite natural number. Clearly $Z\cap\mu=0$ so that Z imbeds naturally in $\mathbf{Z}={}^*Z/\mu$. By results of Robinson [**3**, p. 109] on completions of metric spaces, $\mathbf{Z}$ is the completion of Z with respect to the "natural" topology and hence is the ring of ν!-adic integers [**1**]. Similarly, let p be a standard prime and set $\mu_p=\bigcap p^n\cdot {}^*Z$ $(n\in N)$. Then μ_p is the monad of 0 for the usual p-adic

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topology on Z and can be characterized as either the set of nonstandard integers divisible by every finite power of p or as the $^*Z$-ideal generated by numbers of the form p^n, n an infinite natural number. Thus $\bar{Z}_p = {}^*Z/\mu_p$ is the ring of p-adic integers. It is not difficult to show that $\mu = \bigcap \mu_p$ $(p \in P)$, and then using the fact that $^*U$ is comprehensive, that $\bar{Z} \simeq \prod \bar{Z}_p$ $(p \in P)$.

4. **The units of $\bar{Z}$.** Robinson [**5**, p. 770] notes that the units of $\bar{Z}$ are the residue classes of nonstandard integers which have no standard prime factors. Using Dirichlet's theorem we can sharpen this result and perhaps shed some light on infinite primes. If $x \in {}^*Z$ we shall let $\bar{x}$ denote its residue class in $\bar{Z}$.

THEOREM. *The units of $\bar{Z}$ are precisely the residue classes $\bar{p}$ where p ranges over the infinite primes.*

PROOF. If $p \in P_\infty$ there is an infinite natural number $n<p$, and hence $n!$ and p are prime. Thus $\mu + p \cdot {}^*Z = {}^*Z$ and $\bar{p}$ is a unit of $\bar{Z}$.

Conversely, if $\bar{a}$ is a unit of $\bar{Z}$, $a \cdot {}^*Z + \mu = {}^*Z$, hence a and b are coprime for some nonzero $b \in \mu$. We may assume a and b are positive and, using Dirichlet's theorem, conclude that $a+bn=p$ is prime for some $n \in {}^*N$. Since b is infinite, so is p, and clearly $\bar{a}=\bar{p}$.

This theorem has an interesting converse which points to a possible nonstandard "elementary" proof of Dirichlet's theorem.

THEOREM. *Assume that the units of $\bar{Z}$ are the residues of infinite primes. Then Dirichlet's theorem holds.*

PROOF. Let a and b be standard coprime integers and consider the sequence $\{a+bn\}$ $(n \in N)$. If k is any standard natural number, there is an $n \in N$ such that $a+bn$ is relatively prime to $k!$ (choose n to be the largest factor of $k!$ that is prime to a). Consequently, if k is an infinite natural number, there is an $n \in {}^*N$ such that $a+bn$ and $k!$ are relatively prime. Then $a+bn$ has no standard prime factor and so (see remark at beginning of this section) $(a+bn)^-$ is a unit in $\bar{Z}$.

We consider two cases:

(i) If $b>0$, $(a+bn)^-$ is a unit in $\bar{Z}$ and, by our assumption, $a+bn=p+d$ for some infinite prime p and $d \in \mu$. Since b is standard it divides d, and setting $d=bD$ we see that $a+b(n-D)=p$. Since p is positive infinite, $n-D$ must be positive infinite.

(ii) If $b<0$, $(-a-bn)^-$ is a unit in $\bar{Z}$ and by an argument similar to the one above, $-a-b(n+D)=p$ where again $n+D \in {}^*N$. In either case we have $|a+bk|=p$ for some $k \in {}^*N$. Dirichlet's theorem follows from the Lemma.

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Department of Mathematics, Drexel University, Philadelphia, Pennsylvania 19104