PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 1, May 1974

INJECTIVE VON NEUMANN ALGEBRAS

RICHARD I. LOEBL

ABSTRACT. Injective von Neumann algebras are defined, and a characterization of them as complemented subspaces of $\mathscr{L}(H)$ is given. Several examples and applications are discussed.

The following observation shows an interesting relationship between the geometric structure of von Neumann algebras, i.e., the existence of projections of norm 1 onto them, and their extension properties as ranges of completely positive maps. Since this remark has some applications, e.g. [7], we wish to bring it to more widespread attention.

Following Stinespring [6], we say that a continuous linear map φ between two C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ is *completely positive* if $\forall n \geqq 1$, the natural map φ_n from the C^*-algebra $\mathscr{A} \otimes M_n$ to the C^*-algebra $\mathscr{B} \otimes M_n$ is positive, where M_n is the C^*-algebra of $n \times n$ matrices over the complex numbers.

We say that a von Neumann algebra $\mathscr{R}$ is *injective* if the following condition holds: for every C^*-algebra $\mathscr{B}$, for every selfadjoint linear subspace S of B, containing the identity e of $\mathscr{B}$, and for every completely positive linear map $\varphi: S \rightarrow \mathscr{R}$, there is a completely positive linear map $\psi: \mathscr{B} \rightarrow \mathscr{R}$ such that $\psi_{1|_s} = \varphi$.

Arveson [2] has shown that $\mathscr{L}(H)$, the von Neumann algebra of all bounded linear operators on a Hilbert space H, is injective; and further that, for any range algebra, such an extension is norm-preserving, since $\|\varphi\| = \|\varphi(e)\| = \|\varphi_1(e)\| = \|\varphi_1\|$.

LEMMA. *Let $\mathscr{R}$ be a von Neumann subalgebra of $\mathscr{L}(H)$, and let $P: \mathscr{L}(H) \rightarrow \mathscr{R}$ be such that $\|P\| = 1$, $P_{|\mathscr{R}} = \mathrm{id}$; that is, P is a Banach space projection of norm 1. Then P is completely positive.*

PROOF. The Lemma follows from Tomiyama [8].

THEOREM. *A von Neumann subalgebra $\mathscr{R}$ of $\mathscr{L}(H)$ is injective iff there exists $P: \mathscr{L}(H) \rightarrow \mathscr{R}$, P a Banach space projection of norm 1.*

Received by the editors March 16, 1973.

AMS (MOS) subject classifications (1970). Primary 46L10, 47D15; Secondary 46L05, 22D15.

Key words and phrases. Von Neumann algebras, injective, completely positive maps, property P, Banach space projection of norm one.

PROOF. Suppose $\mathscr{R}$ is injective; then considering $\mathscr{R}$ as a norm-closed selfadjoint subspace containing the identity of $\mathscr{L}(H)$, the completely positive map $\mathrm{id}:\mathscr{R}\to\mathscr{R}$ has an extension $P:\mathscr{L}(H)\to\mathscr{R}$. Clearly, P is the desired projection.

Conversely, let $\mathscr{B}$ be any C^*-algebra, $S\subset\mathscr{B}$ a norm-closed selfadjoint subspace containing the identity of $\mathscr{B}$, and let $\varphi:S\to\mathscr{R}$ be a completely positive map. Let $\mathrm{id}:\mathscr{R}\to\mathscr{L}(H)$ be the natural injection; then id is completely positive, and so is $\hat{\varphi}=\mathrm{id}\circ\varphi:S\to\mathscr{L}(H)$. By the results of Arveson [2], $\hat{\varphi}$ has a completely positive extension $\hat{\psi}:\mathscr{B}\to\mathscr{L}(H)$. Now, letting P be a norm 1 projection onto $\mathscr{R}$, consider the map $\psi=P\circ\hat{\psi}$. By our lemma, P is completely positive, hence so is ψ. Also, ψ clearly extends φ. Q.E.D.

REMARKS. (1) Hakeda and Tomiyama [4] say that a von Neumann algebra has the *extension property* if there exists a Banach space projection of norm 1 from $\mathscr{L}(H)$ onto $\mathscr{R}$. Hence, our result says that a von Neumann algebra is injective iff it has the extension property. It then follows from the work of Tomiyama [7] that $\mathscr{R}$ is injective iff $\mathscr{R}'$ is.

(2) The reader may note the similarity between the idea of injective von Neumann algebras and $\mathscr{B}_1$-Banach spaces [3].

(3) We could have defined injective C^*-algebras, and it would follow as before that a C^*-algebra $\mathscr{A}$ is injective iff, considered as imbedded in $\mathscr{L}(H)$, there is a projection of norm 1 from $\mathscr{L}(H)$ onto $\mathscr{A}$. However, we feel that for applications the von Neumann algebra case is more important. One reason for this is a result of Akemann [1] that the only separable injective C^*-algebras (or von Neumann algebras) are finite dimensional, thus excluding most singly-generated operator algebras.

(4) It is a consequence of the results of Schwartz [5] that if a von Neumann algebra $\mathscr{R}$ has property P, then $\mathscr{R}'$ is injective. Schwartz showed that every hyperfinite factor has property P. Combining this result with one of Tomiyama [7], we have

COROLLARY. *Every hyperfinite factor is injective.*

(5) Similarly, it follows from [4] and [7] that every Type I von Neumann algebra is injective. Hence the class of injective von Neumann algebras includes many interesting examples. However, by a modification of the argument in [5], it follows that there exists a factor of Type II_1 which is not injective [7].

Question. Let $\mathscr{A}$ be an injective C^*-algebra. Is $\mathscr{A}''$ injective?

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Department of Mathematics, University of California, Santa Cruz, California 95060

Current address: Department of Mathematics, Wayne State University, Detroit, Michigan 48202