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ARENS MULTIPLICATION AND A CHARACTERIZATION OF W^*-ALGEBRAS[1]

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ABSTRACT. Let $\mathfrak{A}$ be a Banach algebra which is the dual of a normed linear space $\mathfrak{X}$. Suppose the multiplication in $\mathfrak{A}$ is a continuous function of each factor separately in the weak* topology. We show that the natural projection of $\mathfrak{A}^{**}=\mathfrak{X}^{***}$ onto $\mathfrak{A}=\mathfrak{X}^*$ is a homomorphism with respect to either Arens' multiplication. From this we derive a simple proof of a variant form of Sakai's characterization of W^*-algebras.

Introduction. S. Sakai [6] has shown that a B^*-algebra is a W^*-algebra if and only if it is isometrically linearly isomorphic to the dual of some Banach space. The proof is long and complicated [6], [7]. J. Tomiyama [8] provided an attractive proof which relies on the canonical contractive projection of the triple dual of a Banach space onto the first dual. Both Sakai's proof and Tomiyama's proof depend heavily on the structure of B^*-algebras and on the isometric or contractive nature of the maps involved.

R. Arens [1], [2] defined two products on the double dual of a normed algebra. In this note we show that any Banach algebra which is homeomorphically linearly isomorphic to the dual of a normed linear space and in which multiplication is a weak* continuous function of each factor separately has a continuous homomorphism of its double dual algebra with either Arens' multiplication onto the original algebra. This easy result, which we actually state in a slightly different form, provides a simple proof of the following characterization of W^*-algebras among B^*-algebras:

A B^*-algebra $\mathfrak{A}$ is a W^*-algebra if and only if there is a linear manifold Γ in $\mathfrak{A}^*$ such that: (a) *Each norm continuous linear functional on Γ is defined by evaluation at a unique element of $\mathfrak{A}$*; *and* (b) *the functions $a \to a^*$, $a \to ab$ and $a \to ba$ are continuous in the weak topology on $\mathfrak{A}$ defined by Γ for each $b \in \mathfrak{A}$.*

This elementary characterization of W^*-algebras seems to be as easy to apply as Sakai's much deeper characterization. Sakai's characterization

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can be derived from this one by proving that the continuity of the involution and of multiplication in each factor separately in the weak* topology is automatic when a B^*-algebra is isometrically linearly isomorphic to the dual of some Banach space. The proof of the continuity of these functions is a major portion of the usual proof of Sakai's characterization.

Results. We give the definitions of Arens' two products on the double dual of a normed algebra in three stages. Let $\mathfrak{A}$ be a normed algebra, and let $\pi:\mathfrak{A}\to\mathfrak{A}^{**}$ be the natural injection.

For $a \in \mathfrak{A}$ and $\omega \in \mathfrak{A}^*$ let

$$ {}_a\omega(b) = \omega(ba) \quad \text{and} \quad \omega_a(b) = \omega(ab) \qquad \forall b \in \mathfrak{A}. \tag{1} $$

Then ${}_a\omega$ and ω_a are both elements of $\mathfrak{A}^*$ which depend linearly and continuously on a and ω, and satisfy $\|{}_a\omega\| \leqq \|a\|\,\|\omega\|$, $\|\omega_a\| \leqq \|a\|\,\|\omega\|$, ${}_a({}_c\omega) = {}_{ac}\omega$, $(\omega_a)_c = \omega_{ac}$ and ${}_c(\omega_a) = ({}_c\omega)_a$ for all $\omega \in \mathfrak{A}^*$ and $a, c \in \mathfrak{A}$.

For $f \in \mathfrak{A}^{**}$ and $\omega \in \mathfrak{A}^*$ let

$$ {}_f\omega(a) = f(\omega_a) \quad \text{and} \quad \omega_f(a) = f({}_a\omega) \qquad \forall a \in \mathfrak{A}. \tag{2} $$

Then ${}_f\omega$ and ω_f are both elements of $\mathfrak{A}^*$ which depend linearly and continuously on ω and f and satisfy $\|{}_f\omega\| \leqq \|f\|\,\|\omega\|$, $\|\omega_f\| \leqq \|f\|\,\|\omega\|$, $({}_f\omega)_a = {}_f(\omega_a)$, ${}_a(\omega_f) = ({}_a\omega)_f$, ${}_{\pi(a)}\omega = {}_a\omega$ and $\omega_{\pi(a)} = \omega_a$ for all $f \in \mathfrak{A}^{**}$, $\omega \in \mathfrak{A}^*$ and $a \in \mathfrak{A}$.

For $f, g \in \mathfrak{A}^{**}$ let

$$ fg(\omega) = f({}_g\omega), \qquad f\cdot g(\omega) = g(\omega_f) \qquad \forall \omega \in \mathfrak{A}^*. \tag{3} $$

Then fg and $f\cdot g$ are both elements of $\mathfrak{A}^{**}$, and $\mathfrak{A}^{**}$ is a Banach algebra under either of these multiplications. The linearity in each variable and the submultiplicativity of the norm follow directly from the definition, but before checking the associativity of these multiplications it is convenient to note that ${}_{fg}\omega = {}_f({}_g\omega)$ and $\omega_{f\cdot g} = (\omega_f)_g$ for all $\omega \in \mathfrak{A}^*$ and $f, g \in \mathfrak{A}^{**}$.

It is easy to check that $fg = f\cdot g$ if either f or g is in the image, $\pi(\mathfrak{A})$, of $\mathfrak{A}$ in $\mathfrak{A}^{**}$. However, in general the two multiplications need not agree. It is also immediate that $\pi:\mathfrak{A}\to\mathfrak{A}^{**}$ is a homomorphism; and that if $\mathfrak{A}$ has a multiplicative identity 1 then $\pi(1)$ is a multiplicative identity for $\mathfrak{A}^{**}$. Finally we remark that $f\to fg$ and $g\to f\cdot g$ are both continuous functions in the $\mathfrak{A}^*$ (i.e. weak*) topology on $\mathfrak{A}^{**}$. These assertions are all easily checked.

If $\mathfrak{A}$ is a normed algebra with a continuous involution (i.e. a normed *-algebra in which the involution is continuous) then we will define conjugate linear, involutive maps $*:\mathfrak{A}^*\to\mathfrak{A}^*$ and $*:\mathfrak{A}^{**}\to\mathfrak{A}^{**}$ such that

$$ (fg)^* = g^*\cdot f^* \quad \text{and} \quad (f\cdot g)^* = g^*f^* \qquad \forall f, g \in \mathfrak{A}^{**}. \tag{4} $$

For $\omega \in \mathfrak{A}^*$ let

$$\omega^*(a) = \omega(a^*)^- \qquad \forall a \in \mathfrak{A}. \tag{5}$$

Then ω^* belongs to $\mathfrak{A}^*$, and $*:\mathfrak{A}^* \to \mathfrak{A}^*$ has the properties just asserted and also satisfies $({}_a\omega)^* = (\omega^*)_{a^*}$ for all $a \in \mathfrak{A}$ and $\omega \in \mathfrak{A}^*$. For all $f \in \mathfrak{A}^{**}$ let

$$f^*(\omega) = f(\omega^*)^- \qquad \forall \omega \in \mathfrak{A}^*. \tag{6}$$

Then f^* belongs to $\mathfrak{A}^{**}$, and $*:\mathfrak{A}^{**} \to \mathfrak{A}^{**}$ has the properties claimed above including (4). Before checking (4) it is convenient to note that $({}_f\omega)^* = (\omega^*)_{f^*}$ for all $f \in \mathfrak{A}^{**}$ and $\omega \in \mathfrak{A}^*$. If $fg = f \cdot g$ for all $f, g \in \mathfrak{A}^{**}$ then $*$ is an involution on $\mathfrak{A}^{**}$ and $\pi:\mathfrak{A} \to \mathfrak{A}^{**}$ is a $*$-homomorphism.

If Γ is a linear subspace of the dual of a normed linear space $\mathfrak{X}$, we will call the weak topology on $\mathfrak{X}$ defined by Γ the Γ-topology, and will use such terms as Γ-continuous, etc.

THEOREM 1. *Let $\mathfrak{A}$ be a Banach algebra and let Γ be a linear subspace of $\mathfrak{A}^*$ such that*

(a) *Every norm continuous linear functional on Γ has the form $\pi(a)$ for a unique $a \in \mathfrak{A}$.*

(b) *The functions $a \to ab$ and $a \to ba$ are both Γ-continuous functions of $a \in \mathfrak{A}$ for each fixed $b \in \mathfrak{A}$.*

For each $f \in \mathfrak{A}^{**}$ let $\varphi(f)$ be the unique element of $\mathfrak{A}$ such that $f(\tau) = \tau(\varphi(f))$ for all $\tau \in \Gamma$. Then

$$\varphi(fg) = \varphi(f)\varphi(g) = \varphi(f \cdot g) \qquad \forall f, g \in \mathfrak{A}^{**}. \tag{7}$$

If $*$ is an involution on $\mathfrak{A}$ which is both norm continuous and Γ-continuous then

$$\varphi(f^*) = \varphi(f)^* \qquad \forall f \in \mathfrak{A}^{**}. \tag{8}$$

PROOF. Notice that Γ is a total linear subspace of $\mathfrak{A}^*$. For $\tau \in \Gamma$ and $a \in \mathfrak{A}$, ${}_a\tau$ and τ_a are Γ-continuous by the Γ-continuity of multiplication. Hence by a standard, elementary result [**4**, V. 3.9], ${}_a\tau$ and τ_a belong to Γ. Thus for any $f \in \mathfrak{A}^{**}$, $\tau \in \Gamma$ and $a \in \mathfrak{A}$, ${}_f\tau(a) = f(\tau_a) = \tau_a(\varphi(f)) = \tau(a\varphi(f)) = {}_{\varphi(f)}\tau(a)$ so ${}_f\tau = {}_{\varphi(f)}\tau$. Similarly $\tau_f = \tau_{\varphi(f)}$ for all $f \in \mathfrak{A}^{**}$ and $\tau \in \Gamma$. Now for any $f, g \in \mathfrak{A}^{**}$ and $\tau \in \Gamma$,

$$\tau(\varphi(fg)) = fg(\tau) = f({}_g\tau) = f({}_{\varphi(g)}\tau) = {}_{\varphi(g)}\tau(\varphi(f)) = \tau(\varphi(f)\varphi(g)),$$

so $\varphi(fg) = \varphi(f)\varphi(g)$. Similarly $\varphi(f \cdot g) = \varphi(f)\varphi(g)$ for all $f, g \in \mathfrak{A}^{**}$.

When $*$ is norm continuous, τ^* is defined for each $\tau \in \Gamma$ and when $*$ is Γ-continuous, τ^* is Γ-continuous and hence belongs to Γ. Thus for $f \in \mathfrak{A}^{**}$ and $\tau \in \Gamma$, $\tau(\varphi(f^*)) = f^*(\tau) = f(\tau^*)^- = \tau^*(\varphi(f))^- = \tau(\varphi(f)^*)$, so $\varphi(f^*) = \varphi(f)^*$.

REMARK 1. Notice that $\varphi \circ \pi$ is the identity map on $\mathfrak{A}$ and consequently $\pi \circ \varphi$ is a projection of $\mathfrak{A}^{**}$ onto $\pi(\mathfrak{A})$.

REMARK 2. If we assume only that $a \to ab$ is a Γ-continuous function of $a \in \mathfrak{A}$ for each $b \in \mathfrak{A}$, then we can conclude that: (1) ${}_a\tau \in \Gamma$ for each $a \in \mathfrak{A}$ and $\tau \in \Gamma$; (2) $\tau_f = \tau_{\varphi(f)}$ for each $f \in \mathfrak{A}^{**}$ and $\tau \in \Gamma$; (3) $\varphi(f\pi(a)) = \varphi(f)a = \varphi(f \cdot \pi(a))$ for each $f \in \mathfrak{A}^{**}$ and $a \in \mathfrak{A}$. Analogous results hold for the Γ-continuity of left multiplication.

REMARK 3. The involution on $\mathfrak{A}$ is Γ-continuous if and only if Γ is invariant under the map $*: \mathfrak{A}^* \to \mathfrak{A}^*$ defined by equation (5) above. It may be easier to check this alternate formulation in some cases.

From now on we will assume that all algebras and linear spaces have complex scalars. The foregoing results and arguments are correct with real or complex scalars provided only that "conjugate linear" is interpreted to mean "linear" in the real case. However the following considerations must be reinterpreted in less trivial ways in order to apply to the real case.

A B^*-algebra is a $*$-algebra which has a faithful $*$-representation as a C^*-algebra (i.e. a norm closed $*$-subalgebra of the $*$-algebra $B(\mathfrak{H})$ of bounded operators on some Hilbert space $\mathfrak{H}$). A B^*-algebra carries the ($*$-algebraically defined) norm, $\|a\| = \rho(a^*a)^{1/2}$ where ρ is the spectral radius. Relative to this norm a B^*-algebra is a Banach $*$-algebra with an isometric involution, all its $*$-representations are contractive and all its faithful $*$-representations are isometric. A W^*-algebra is a $*$-algebra which has a faithful $*$-representation as a von Neumann algebra (i.e. a $*$-subalgebra $\mathfrak{A}$ of some $B(\mathfrak{H})$ such that $\mathfrak{A}$ equals its own double commutant $\mathfrak{A}''$).

For any Hilbert space $\mathfrak{H}$ and any $x, y \in \mathfrak{H}$ we define

$$\omega_{x,y}(T) = (Tx, y) \qquad \forall T \in \mathfrak{B}(\mathfrak{H}). \tag{9}$$

Then $\omega_{x,y}$ is a linear functional on $\mathfrak{B}(\mathfrak{H})$. The weak topology on $\mathfrak{B}(\mathfrak{H})$ defined by the linear span of the set $\{\omega_{x,y} : x, y \in \mathfrak{H}\}$ is the weak operator topology. Since multiplication in $\mathfrak{B}(\mathfrak{H})$ is a continuous function of each factor separately in the weak operator topology it is easy to see that any von Neumann algebra is closed in the weak operator topology. We can now state the three well known results which we need. Easy elementary proofs are contained in **[5]**.

LEMMA 1 (VON NEUMANN DENSITY THEOREM). *Let $\mathfrak{H}$ be a Hilbert space and let $\mathfrak{A}$ be a $*$-subalgebra of $\mathfrak{B}(\mathfrak{H})$ which is closed in the weak operator topology. Then $\mathfrak{A}$ contains a multiplicative identity E which is a projection in $\mathfrak{B}(\mathfrak{H})$. The set $\{T|E\mathfrak{H} : T \in \mathfrak{A}\}$ of operators in $\mathfrak{A}$ restricted to $E\mathfrak{H}$ is a von Neumann algebra on $E\mathfrak{H}$. If $\mathfrak{A}$ is essential (i.e. $\{Tx : T \in \mathfrak{A}, x \in \mathfrak{H}\}$ is dense in $\mathfrak{H}$) then E is the identity operator.*

LEMMA 2. *Let $\mathfrak{A}$ be a B^*-algebra. Then there is a faithful, essential $*$-representation Υ of $\mathfrak{A}$ such that every linear functional ω on $\mathfrak{A}$ can be written as $\omega_{x,y}\circ\Upsilon$ for $x, y \in \mathfrak{H}^\Upsilon$ with $\|x\|\,\|y\| \leqq 2\|\omega\|$.*

The representation Υ of Lemma 2 can be chosen to be the universal representation which is the Hilbert sum of all the $*$-representations corresponding to states of $\mathfrak{A}$. The constant 2 in Lemma 2 can be replaced by 1 but this is harder to prove, and is unnecessary for our purposes.

The next well-known result [**3**, 7.1], has an easy proof using only Lemma 2 and elementary considerations. We give this proof below.

LEMMA 3. *Let $\mathfrak{A}$ be a B^*-algebra and let Υ be a $*$-representation satisfying Lemma 2. For each $f\in\mathfrak{A}^{**}$ there is a unique operator $\overline{\Upsilon}_f$ in $\mathfrak{B}(\mathfrak{H}^\Upsilon)$ such that*

$$(\overline{\Upsilon}_f x, y) = f(\omega_{x,y}\circ\Upsilon) \qquad \forall x, y \in \mathfrak{H}^\Upsilon.$$

Furthermore:

(a) *The two Arens multiplications agree so that $\mathfrak{A}^{**}$ is a $*$-algebra.*

(b) *$\overline{\Upsilon}$ is a $*$-isomorphism onto the von Neumann algebra generated by $\Upsilon_\mathfrak{A}$.*

(c) *$\overline{\Upsilon}$ is a homeomorphism with respect to the $\mathfrak{A}^*$-topology on $\mathfrak{A}^{**}$ and the weak operator topology on $\mathfrak{B}(\mathfrak{H}^\Upsilon)$.*

(d) *$\overline{\Upsilon}\circ\pi=\Upsilon$.*

REMARK. It is also true that $\overline{\Upsilon}$ is an isometry. This can be proved by a simple application of the Kaplansky density theorem, but we omit this since we do not need the result.

PROOF. The expression $f(\omega_{x,y}\circ\Upsilon)$ depends linearly on x, conjugate linearly on y and satisfies $|f(\omega_{x,y}\circ\Upsilon)|\leqq\|f\|\,\|\omega_{x,y}\circ\Upsilon\|\leqq\|f\|\,\|x\|\,\|y\|$. Hence there is a unique operator $\overline{\Upsilon}_f\in\mathfrak{B}(\mathfrak{H})$ such that $(\overline{\Upsilon}_f x, y)=f(\omega_{x,y}\circ\Upsilon)$ for all $x, y\in\mathfrak{H}^\Upsilon$. Obviously $\overline{\Upsilon}:\mathfrak{A}^{**}\to\mathfrak{B}(\mathfrak{H}^\Upsilon)$ is a linear contraction, and $\overline{\Upsilon}\circ\pi=\Upsilon$. If $\overline{\Upsilon}_f=0$ then $f(\omega_{x,y}\circ\Upsilon)=0$ for all $x, y\in\mathfrak{H}^\Upsilon$ which implies $f=0$ by Lemma 2. Hence $\overline{\Upsilon}$ is injective. Since each linear functional on $\mathfrak{A}$ has the form $\omega_{x,y}\circ\Upsilon$ for some $x, y\in\mathfrak{H}^\Upsilon$, $\overline{\Upsilon}$ is a homeomorphism from the $\mathfrak{A}^*$-topology to the weak operator topology.

It is easy to check that ${}_a(\omega_{x,y}\circ\Upsilon)=\omega_{\Upsilon_a x,y}\circ\Upsilon$ and $(\omega_{x,y}\circ\Upsilon)_a=\omega_{x,\Upsilon_a{}^*y}\circ\Upsilon$ for all $x, y\in\mathfrak{H}^\Upsilon$ and $a\in\mathfrak{A}$. Hence ${}_f(\omega_{x,y}\circ\Upsilon)=\omega_{x,\overline{\Upsilon}_f{}^*y}\circ\Upsilon$ and $(\omega_{x,y}\circ\Upsilon)_f=\omega_{\overline{\Upsilon}_f x,y}\circ\Upsilon$ for all $x, y\in\mathfrak{H}^\Upsilon$ and $f\in\mathfrak{A}^{**}$. From this it follows immediately that $\overline{\Upsilon}_{fg}=\overline{\Upsilon}_f\overline{\Upsilon}_g=\overline{\Upsilon}_{f\cdot g}$ and $\overline{\Upsilon}_{f^*}=\overline{\Upsilon}_f^*$ for all f, $g\in\mathfrak{A}^{**}$.

Since Υ is continuous from the $\mathfrak{A}^*$-topology to the weak operator topology and since $\pi(\mathfrak{A})$ is $\mathfrak{A}^*$-dense in $\mathfrak{A}^{**}$ [**4**, V. 4.5], $\Upsilon_{\mathfrak{A}^{**}}$ is included in the weak operator closure of $\overline{\Upsilon}_{\pi(\mathfrak{A})}=\Upsilon_\mathfrak{A}$. Conversely if S belongs to the weak operator closure of $\Upsilon_{\mathfrak{A}}$ define $\hat{S}$ by $\hat{S}(\omega_{x,y} \circ \Upsilon)=(Sx, y)$. Approximating S in the weak operator topology by a net of elements in $\Upsilon_{\mathfrak{A}}$ shows that $\hat{S}$ is a well defined element of $\mathfrak{A}^{**}$. It is obvious that $\overline{\Upsilon}_{\hat{S}}=S$. This concludes the proof of the lemma.

THEOREM 2. *Let $\mathfrak{A}$ be a B^*-algebra. Then $\mathfrak{A}$ is a W^*-algebra if and only if there is a linear subspace Γ of $\mathfrak{A}^*$ such that:*

(a) *Every norm continuous linear functional on Γ has the form $\pi(a)$ for a unique $a \in \mathfrak{A}$.*

(b) *The multiplication of $\mathfrak{A}$ is a Γ-continuous function of each factor separately, and the involution is Γ-continuous.*

PROOF. Suppose $\mathfrak{A}$ is a W^*-algebra. Let T be a faithful $*$-representation of $\mathfrak{A}$ as a von Neumann algebra on a Hilbert space $\mathfrak{H}$. Let Γ be the linear span of the functionals $\omega_{x,y} \circ T$ for all $x, y \in \mathfrak{H}$.

Let f be a norm continuous linear functional on Γ. Then $\langle x, y\rangle= f(\omega_{x,y} \circ T)$ depends linearly on x, conjugate linearly on y and satisfies $|\langle x, y\rangle| \leqq \|f\| \, \|\omega_{x,y} \circ T\| \leqq \|f\| \, \|x\| \, \|y\|$ since $\|T_a\| \leqq \|a\|$. Thus there is a unique $S \in \mathfrak{B}(\mathfrak{H})$ such that $\langle x, y\rangle=(Sx, y)$. Suppose $W \in T'_{\mathfrak{A}}$. Then $\omega_{Wx,y} \circ T=\omega_{x,W^*y} \circ T$ so $(SWx, y)=(Sx, W^*y)=(WSx, y)$ for all x, $y \in \mathfrak{H}$. Hence S commutes with W. Therefore $S \in T''_{\mathfrak{A}}=T_{\mathfrak{A}}$. Thus there is a unique $a \in \mathfrak{A}$ such that $\langle x, y\rangle=(T_a x, y)$. Hence $f(\omega_{x,y} \circ T)=\langle x, y\rangle= (T_a x, y)=\omega_{x,y}(T_a)=\pi(a)(\omega_{x,y} \circ T)$ for all $x, y \in \mathfrak{H}$. Hence Γ satisfies (a).

The Γ-topology on $\mathfrak{A}$ is the weak operator topology on $T_{\mathfrak{A}}$ pulled back through T. Thus it is well known, (and elementary to check) that (b) holds. This proves the necessity of the condition.

Now suppose a linear subspace $\Gamma \in \mathfrak{A}^*$ satisfying the stated conditions is given. By Lemma 3(a), $*:\mathfrak{A}^{**}\rightarrow\mathfrak{A}^{**}$ is an involution. Hence the homomorphism $\varphi:\mathfrak{A}^{**}\rightarrow\mathfrak{A}$ guaranteed by Theorem 1 is a $*$-homomorphism. Let $\mathfrak{J}$ be the kernel of φ. Then the definition of φ shows that $\mathfrak{J}$ is closed in the $\mathfrak{A}^*$ (i.e. weak*) topology on $\mathfrak{A}^{**}$.

Let $\overline{\Upsilon}$ be the faithful $*$-representation of $\mathfrak{A}^{**}$ as a von Neumann algebra described in Lemma 3. Since $\overline{\Upsilon}$ is faithful there is an identity element 1 for $\mathfrak{A}^{**}$. Since $\overline{\Upsilon}$ is a homeomorphism from the $\mathfrak{A}^*$ topology on $\mathfrak{A}^{**}$ to the weak operator topology on $\mathfrak{B}(\mathfrak{H}^{\Upsilon})$, $\overline{\Upsilon}_{\mathfrak{J}}$ is closed in the weak operator topology. Hence by Lemma 1 there is a projection $E \in \mathfrak{B}(\mathfrak{H}^{\Upsilon})$ such that $E \in \overline{\Upsilon}_{\mathfrak{J}}$ is an identity for $\overline{\Upsilon}_{\mathfrak{J}}$. Let $e \in \mathfrak{J}$ be the (unique) element of $\mathfrak{J}$ such that $\overline{\Upsilon}_e=E$. Since $\overline{\Upsilon}$ is faithful, e is an identity for $\mathfrak{J}$. Hence $\mathfrak{A}^{**}e \subseteq \mathfrak{J} \subseteq e\mathfrak{J}e \subseteq e\mathfrak{A}^{**}e \subseteq \mathfrak{A}^{**}e$. Thus for any $f \in \mathfrak{A}^{**}$, $fe=efe=(ef^*e)^*=(f^*e)^*=ef$, so e is a central projection. Therefore the restriction of φ to $\mathfrak{A}^{**}(1-e)$ is a $*$-isomorphism onto $\mathfrak{A}$. Call the inverse ψ. For each $a \in \mathfrak{A}$ let $T_a= \overline{\Upsilon}_{\psi(a)}|(I-E)\mathfrak{H}^{\Upsilon}$. Then $T_{\mathfrak{A}}$ is a weakly closed $*$-subalgebra of $\mathfrak{B}((I-E)\mathfrak{H}^{\Upsilon})$

containing the identity. Thus T is a faithful *-representation of $\mathfrak{A}$ as a von Neumann algebra. Hence $\mathfrak{A}$ is a W^*-algebra.

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