

THE ESSENTIAL SPECTRUM OF SOME TOEPLITZ OPERATORS

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ABSTRACT. The localization techniques of Douglas and Sarason are used to obtain the essential spectrum of the Toeplitz operator T_φ for which φ is the product of a continuous function and the characteristic function of a measurable subset of the unit circle. Examples are given of Toeplitz operators with one-dimensional self-commutator whose essential spectrum is the unit disk. Using an example of J. E. Brennan, the authors show the existence of a completely nonnormal, subnormal operator whose adjoint has no point spectrum.

Introduction. Let μ denote normalized Lebesgue measure on the unit circle T and let $L^p(\mu) = L^p$, $1 \leq p \leq \infty$, denote the complex Lebesgue spaces. Let $H^p \subset L^p$ denote the usual Hardy spaces. If P is the orthogonal projection of L^2 onto H^2 and if $\varphi \in L^\infty$, then the Toeplitz operator with symbol φ , denoted T_φ , is defined by $T_\varphi(f) = P(\varphi f)$ for $f \in H^2$. Toeplitz operators have been studied extensively in the past two decades. The interested reader will find a well-written discussion of the current "state of the art" for Toeplitz operators in [5].

Recall that if the symbol φ is continuous, then the essential spectrum of T_φ is the range of φ . In case φ is only piecewise continuous, the essential spectrum of T_φ is the curve $\varphi^\#$ formed by taking the union of the range of φ (range always means essential range) and the line segments joining $\varphi(t+)$ and $\varphi(t-)$ at any point of discontinuity t of φ . This note describes the essential spectrum of T_φ in case $\varphi = f\chi_E$, where f is continuous and χ_E is the characteristic function of a measurable subset E of the unit circle. Our description of the essential spectrum (in Theorem 3 of §2) should be viewed as an application of the localization result of Douglas and Sarason [6].

In §3, we use the results of §2 to construct a completely nonnormal hyponormal Toeplitz operator with self-commutator of rank one whose essential spectrum is the closed unit disk.

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Finally, in §4, we discuss the existence of point spectrum for the adjoint of a hyponormal operator. We use an example of Brennan [1] to exhibit a nonnormal, subnormal operator whose adjoint has no point spectrum.

1. Localization. Let X denote the maximal ideal space of L^∞ and recall that X has a natural fibration over the unit circle. For every α of modulus 1, the fibre over α is the set X_α of all homomorphisms of X that assign the value α to the function $\chi(t) = e^{it}$. If φ is in L^∞ and $\hat{\varphi}$ denotes the Gelfand transform of φ , then $\hat{\varphi}(X_\alpha)$ consists of all complex λ which are in the essential range of $\varphi|N(\alpha)$ for every neighborhood $N(\alpha)$ of α (cf. Hoffman [9, p. 171]).

An operator T on a Hilbert space \mathcal{H} is called Fredholm (left-Fredholm) in case T has an inverse (left-inverse) modulo the ideal of compact operators on \mathcal{H} . The essential spectrum of T is the set

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}.$$

The following localization result is due to Douglas and Sarason [6]:

THEOREM 1. *Let φ be a unimodular function in L^∞ . Then T_φ is left-Fredholm if and only if $\max_{|\alpha|=1} \text{dist}(\hat{\varphi}|X_\alpha, \hat{H}^\infty|X_\alpha) < 1$.*

If f is continuous and E is a measurable subset of T , the range of $(f\chi_E)^\wedge$ restricted to each fibre contains at most two points. In this case we shall apply the following theorem [3]:

THEOREM 2. *Suppose φ is a unimodular function in L^∞ such that for some α on the unit circle, $\hat{\varphi}(X_\alpha)$ is a pair of antipodal points. Then $\text{dist}(\hat{\varphi}|X_\alpha, \hat{H}^\infty|X_\alpha) = 1$.*

Actually, Theorems 1 and 2 may be used to describe the essential spectrum of the Toeplitz operator T_φ , if, for every $\alpha \in T$, $\hat{\varphi}(X_\alpha)$ is contained in a line segment L_α . This result has been observed independently by R. G. Douglas. In this note, however, we shall have no need of this generality.

2. The essential spectrum. If E is a measurable subset of the unit circle, then \bar{E} will always denote the essential closure of E , i.e. the set of all x such that $\mu((x-\varepsilon, x+\varepsilon) \cap E) > 0$ for all $\varepsilon > 0$. The main result of this section is:

THEOREM 3. *Let f be continuous on the unit circle and let E be a measurable subset of T , $\mu(E) \neq 0$. Set $\varphi = f\chi_E$. Then*

$$\sigma_e(T_\varphi) = \varphi^+ = \{f(t) : t \in \bar{E}\} \cup \{(1-c)f(t) : t \in \bar{E} \cap \bar{E}^c, 0 \leq c \leq 1\}.$$

PROOF. One of the first results on the invertibility of Toeplitz operators states that if ψ is not invertible in L^∞ , then T_ψ is not Fredholm [6]. This clearly implies that $\{f(t): t \in \bar{E}\} \subset \sigma_e(T_\psi)$. If $\bar{E} \cap \bar{E}' \neq \emptyset$, then E' must have positive measure, so that T_ψ is not Fredholm.

Next, suppose that $0 < c < 1$ and that $t_0 \in \bar{E} \cap \bar{E}'$. We claim that $\lambda = (1-c)f(t_0)$ is in $\sigma_e(T_\psi)$. Obviously, we may assume that $\varphi - \lambda$ is invertible in L^∞ . A result of Devinatz [4] shows that if ψ is invertible in L^∞ , then T_ψ is Fredholm (left-Fredholm) if and only if T_u is Fredholm (left-Fredholm), where $u = \psi/|\psi|$. Consider the function $h_\lambda = (\varphi - \lambda)/|\varphi - \lambda|$. The range of h_λ on the fibre over t_0 consists of $-\lambda/|\lambda| = -f(t_0)/|f(t_0)|$ and $f(t_0)/|f(t_0)|$. An application of Theorems 1 and 2 shows that $T_\psi - \lambda I$ is not Fredholm.

Finally, suppose that $\lambda \neq 0$ is not in φ^+ . Then $\lambda/|\lambda|$ is not in \bar{E} and φ vanishes (almost everywhere) on some neighborhood N of $\lambda/|\lambda|$. Thus, on N , $\varphi - \lambda = -\lambda$ (a.e.). For $\mu \in T$, $\mu \neq \arg \lambda$, there is a neighborhood $N(\mu)$ such that λ is not in the convex hull of the values of $g = \varphi \chi_{N(\mu)}$, and so, $T_g - \lambda I$ is invertible. We have just observed that $T_\psi - \lambda I$ is locally Fredholm, and hence, $T_\psi - \lambda I$ is Fredholm (Douglas and Sarason [6]), completing the proof of Theorem 3.

3. Essential spectrum of hyponormal operators. In this section we use the results of the preceding section to construct examples of hyponormal operators whose essential spectrum has nonempty interior.

Recall that an operator T on a Hilbert space \mathcal{H} is hyponormal if its self-commutator, $[T] = T^*T - TT^*$, is nonnegative.

PROPOSITION 1. *Let S be an isometry of defect n on a Hilbert space \mathcal{H} and suppose that T is an operator on \mathcal{H} which satisfies $T = T^*S$. Then T is hyponormal and $\dim(\text{ran}[T]) \leq n$.*

PROOF. Obviously, $T^* = S^*T$. Therefore,

$$[T] = T^*T - TT^* = T^*T - T^*SS^*T = T^*(I - SS^*)T,$$

which is nonnegative and which has rank at most n .

Recall that if $|\alpha_i| < 1$, $i = 1, 2, \dots, N$, and if

$$B(z) = \prod_{i=1}^N \frac{\alpha_i}{|\alpha_i|} \frac{z - \alpha_i}{1 - \bar{\alpha}_i z}$$

is a finite Blaschke product, then the associated Toeplitz operator T_B is an isometry of defect N . If $\varphi, f \in L^\infty$ and f is real-valued, then $\psi = (\varphi + B\bar{\varphi})f \in L^\infty$ and $\bar{\psi}B = (\bar{\varphi} + \bar{B}\varphi)fB = (\bar{\varphi}B + \varphi)f = \psi$. It follows from Proposition 1 that T_ψ is hyponormal and that $[T_\psi]$ is at most of rank N . Similar computations show that the operators T_φ are hyponormal when

$\varphi = f\chi_E$, where E is a measurable subset of T , and f is any one of the functions: $\eta(t) = e^{it/2}$, $-\pi < t < \pi$, or $\chi^n(t) = e^{int}$, for $n = 1, 2, \dots$

The operator T_η deserves further comment. The self-commutator of T_η is one-dimensional (since $\dot{\eta} = \chi\bar{\eta}$), and the spectrum of T_η is the "half-moon" $M = \{\lambda \in \mathbb{C} : |\lambda| \leq 1, \operatorname{Re} \lambda \geq 0\}$. The real part of T_η is the Toeplitz operator $T_{\cos(t/2)}$. The operator $T_{\cos(t/2)}$ is absolutely continuous with uniform spectral multiplicity equal to one (cf. Rosenblum [10]). It is easily seen that T_η has no reducing subspaces on which T_η is a normal operator. It follows that T_η has a singular integral representation of the form described in [3].

If a measurable subset E of T is chosen so that \bar{E} and \bar{E}' both equal T , and if we let $\psi = \eta\chi_E$, then $\sigma_s(T_\psi) = M$. Although this does not follow directly from Theorem 3 (η is discontinuous), one observes that Theorem 3 is a local result, and this is enough to establish the assertion. The self-commutator of T_ψ is one-dimensional in this case and the real part of T_ψ has uniform spectral multiplicity equal to infinity, so that T_ψ differs considerably from the operators considered in [3].

4. Point spectrum of the adjoint of a hyponormal operator. If a hyponormal operator T is completely nonnormal, i.e., has no reducing subspaces on which T is normal, then the point spectrum of T is empty. The same conclusion does not hold for T^* (consider the unilateral shift). Recall that T on \mathcal{H} is subnormal if there is a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a normal operator N on \mathcal{K} such that $N\mathcal{H} \subseteq \mathcal{H}$ and $N|_{\mathcal{H}} = T$. Every subnormal operator is hyponormal (the inclusion, however, is proper). It follows from a result of Stampfli [11, Theorem 5] that if T is subnormal and has finite rank self-commutator, then the point spectrum of T^* is nonempty. Two questions come to mind. First, if T is subnormal, is the point spectrum of T^* nonempty? Secondly, if T is hyponormal and has finite rank self-commutator, is the point spectrum of T^* nonempty?

It turns out that both of these questions have negative answers. Recently Richard Carey and Joel Pincus [2] announced examples of hyponormal operators with one-dimensional self-commutator whose adjoints have no point spectra. Using the announced results of these authors, we are able to give necessary and sufficient conditions for a complex number λ to be in the point spectra of the hyponormal Toeplitz operators T_ψ , where $\psi = (z+1)\chi_E - 1$. Details of this result will appear elsewhere. Note that these operators T_ψ have a one-dimensional self-commutator. It should be pointed out that when $0 < \mu(E) < 2\pi$, then the kernel of the operator $T_{f\chi_E}$ is zero. This follows since $\ker T_\varphi \neq (0)$ implies $\log|\varphi|$ is integrable. For this and other results on point spectra we refer the interested reader to Hartman [7].

Finally, we show that the first question posed in this section has a negative answer. To this end, let X be a compact subset of the plane and let μ be a positive Baire measure supported on X . Further, let $\mathcal{R}(X)$ denote the algebra of rational functions with poles off X , and let $R^2(d\mu)$ denote the (closed) subspace of $L^2(d\mu)$ spanned by $\mathcal{R}(X)$. The restriction of multiplication by z to $R^2(d\mu)$ will be denoted by A_z . Then A_z is subnormal, and, if the support of μ is X , then $\sigma(A_z) = X$.

In [1], Brennan considers the case in which X is a "Mergelyan Swiss cheese", i.e., a compact, nowhere dense subset of the plane obtained by deleting countably many disjoint open disks D_i whose radii r_i satisfy $\sum_{i=1}^{\infty} r_i < \infty$ from the unit disk. If ds is arc length measure on the union of the unit circle and the boundaries of the disks D_i , then $R^2(ds) \neq L^2(ds)$ (cf. [1, p. 290]).

A closed subspace $M \subset R^2(d\mu)$ will be called $\mathcal{R}(X)$ invariant in case M is invariant under multiplication by every function in $\mathcal{R}(X)$.

The following theorem appears as Theorem 2.6 in [1].

THEOREM 4. *There exists a Swiss cheese X such that $R^2(ds)$ has no proper $\mathcal{R}(X)$ invariant subspaces of finite codimension.*

The following proposition, when combined with Theorem 4, shows that there exist nonnormal, subnormal operators whose adjoint has empty point spectrum.

PROPOSITION 2. *Let X be a compact subset of \mathbb{C} and let μ be a positive Baire measure supported on X . Then $R^2(d\mu)$ has no proper $\mathcal{R}(X)$ invariant subspaces of finite codimension if and only if there are no proper invariant subspaces of finite codimension which are invariant under A_z .*

PROOF. Suppose $M \subset R^2(d\mu)$ is a proper A_z invariant subspace of finite codimension. We may assume, without loss of generality, that $\dim(R^2(d\mu) \ominus M) = 1$. Let P denote the orthogonal projection of $L^2(d\mu)$ onto $R^2(d\mu)$ and suppose that for some $\alpha \notin X$, $(z - \alpha)^{-1}M$ is not a subset of M . Then the restriction of $A_{z-\alpha}$ to M is not invertible and hence, there is an $f \in M$ which is orthogonal to $(z - \alpha)M$. Equivalently, there exists a nonzero $f \in M$ such that $P(\bar{z} - \bar{\alpha})f = h$, where $h \in R^2(d\mu) \ominus M$. Note that $h \neq 0$, since α is not in the spectrum of A_z . Since $\dim(R^2(d\mu) \ominus M) = 1$ and $R^2(d\mu) \ominus M$ is invariant under A_z^* , there exists a β in X such that $P(\bar{z} - \bar{\beta})h = 0$ and so, $P(\bar{z} - \bar{\alpha})P(\bar{z} - \bar{\beta})f = P(\bar{z} - \bar{\beta})P(\bar{z} - \bar{\alpha})f = 0$. Noting again that $\alpha \notin \sigma(A_z)$, we have $P(\bar{z} - \bar{\beta})f = 0$. But $P(\bar{z} - \bar{\beta})f = (\bar{z} - \bar{\beta})f + h$ where f is orthogonal to h , a contradiction, and the proof is complete.

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