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ON ANALYTIC IRREDUCIBILITY AT ∞ OF A PENCIL OF CURVES

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ABSTRACT. In this article we establish that if a member of the pencil $f(x, y)+\lambda$ is analytic irreducible at ∞ then all members are.

1. **Introduction.** In [3] W. Engel used the following statement: "For a special member of the pencil $f(x, y)+\pi=0$ the number of branches at ∞ cannot be greater than the corresponding number for the general one". The above statement has been a main blockade for understanding his proof of the theorem of integral Cremona-transformation or the global Jacobian theorem. In 1971 S. S. Abhyankar disproved it by giving a counterexample (unpublished). However a weaker statement (see Theorem I) can be proved easily by applying the results of [4]. The notations and results of [1], [2], [4] will be used extensively. We shall assume that the ground field k is algebraically closed with characteristic zero.

2. **Theorem I.** Let $f(x, y)$ be a polynomial of degree n defining a curve with only one place at ∞ centered at the infinite point on the x-axis. Let (μ_i) be defined as in [1], [2], [4]. Let $d=\max_i \mu_i/d_i n$. Note that d is a positive number.

THEOREM I. *The curve defined by* $f(x, y)+p(x)$ *with* $\deg p(x)<d$ *has only one place at* ∞. *In particular*, $f(x, y)+c$ *is analytically irreducible at* ∞ *for arbitrary constant* c *in* k.

PROOF. Since $f(x, y)$ defines a curve with one place at ∞, then it follows from Theorem 1 of [4] that there is an ordered factorization of $n=n_1, n_2, \cdots, n_h$ such that the rth power conditions are satisfied $\forall r\leqq h$. Let us use this factorization for $f(x, y)+p(x)$ also. Let $g_r(x, y)$ be the approximate d_rth root of $f(x, y)$, namely $g_r(x, y)$ is the unique polynomial with

$$\deg_y(f(x, y) - g_r(x, y)^{d_r}) < n - (n/d_r).$$

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Clearly

$$\deg_y(f(x, y) + p(x) - g_r(x, y)^{d_r}) < n - (n/d_r).$$

In other words $g_r(x,y)$ is the approximate d_rth root of $f(x,y)+p(x)$. Let us verify the conditions (1), (2) and (3) in Definition 4.2 of [**4**] for $f(x,y)+p(x)$ and the ordered factorization $n=n_1, n_2, \cdots, n_h$. Let

$$f(x, y) = g_r(x, y)^{d_r} + \sum_{i=1}^{d_r}\left(\sum_{\alpha} a_{i\alpha}g^{\alpha}\right)g_r(x, y)^{d_r-i}$$

Clearly

$$f(x, y) + p(x) = g_r(x, y)^{d_r} + \sum_{i=1}^{d_r}\left(\sum_{\alpha} a_{i\alpha}g^{\alpha}\right)g_r(x, y)^{d_r-i} + p(x).$$

The corresponding polynomials in Definition 4.2 of [**4**] for $f(x,y)$ and $f(x,y)+p(x)$, respectively, are as follows

$$y^{d_r} + \sum_{i=1}^{d_r}\left(\sum a_{i\alpha}x^{\alpha}\right)y^{d_r-i} = \sum f_i(x_1, \cdots, x_n)y^{d_r-i},$$

and

$$y^{d_r} + \sum_{i=1}^{d_r}\left(\sum a_{i\alpha}x^{\alpha}\right)y^{d_r-i} + p(x) = \sum f_i(x_1, \cdots, x_n)y^{d_r-i} + p(x).$$

According to the rth power condition for $f(x,y)$, $\mu_r=d_r|\sigma_r f_{d_r}(x_1, \cdots, x_n)|$ is strictly bigger than $|\sigma_r p(x)|=n\deg p(x)$ by our restriction on the degree of $p(x)$. It trivially follows that conditions (1), (2) and (3) in Definition 4.2 in [**4**] are satisfied. Our theorem follows from Theorem I of [**4**]. Q.E.D.

Let $f(x,y)$ be a polynomial defined by a rational curve with one place at ∞; it follows from Theorem I that $f(x,y)+\lambda$ will define the curve with one place at infinity. We propose

CONJECTURE. Let $f(x,y)$ be as previous. The polynomial $f(x,y)+\lambda$ defines rational curves for some nonzero λ iff $f(x,y)$ defines a nonsingular rational curve.

REMARK. After this manuscript had been prepared S. S. Abhyankar showed the author another proof by applying (3.4) of [**1**]. In fact, as pointed out by him, the condition "only one place at ∞" can be replaced by "only one place at point p" without changing either proof.

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